Dispersive models for free-surface flows: structure and numerical strategy

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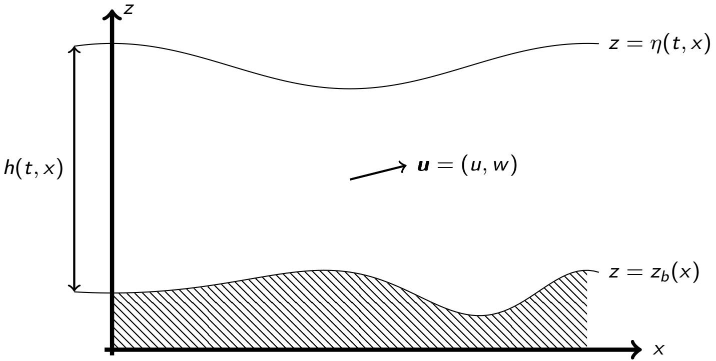
Outline

Fluid domain



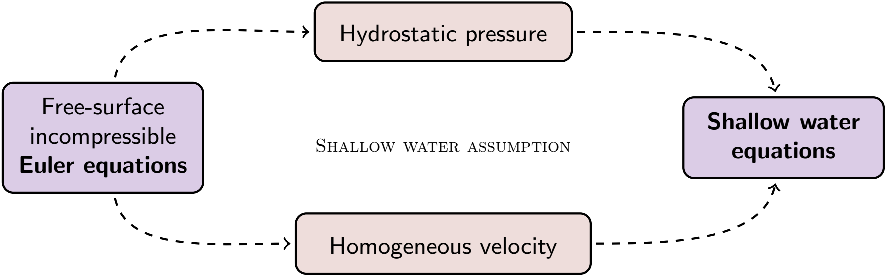
Water height:

$$h(t, x) = \eta(t, x) - z_b(x)$$

Literature about free-surface flows

Free-surface
incompressible
Euler equations

Literature about free-surface flows



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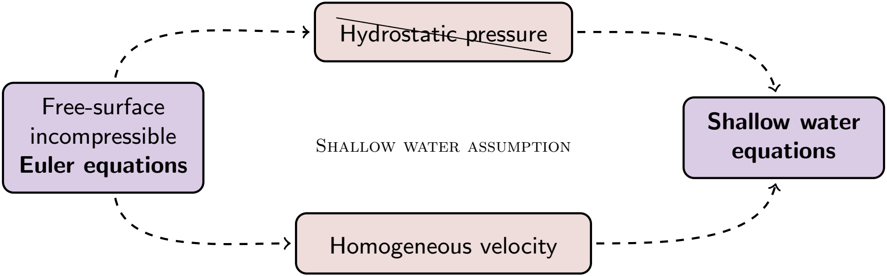
Literature about free-surface flows



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A.E. Green, P.M. Naghdi, *A derivation of equations for wave propagation in water of variable depth* (**J. Fluid Mech.** 78(2), 1976)

M.-O. Bristeau, J. Sainte-Marie, *Derivation of a non-hydrostatic shallow water model; Comparison with Saint-Venant and Boussinesq systems* (**Discrete Contin. Dyn. Syst. Ser. B** 10(4), 2008)

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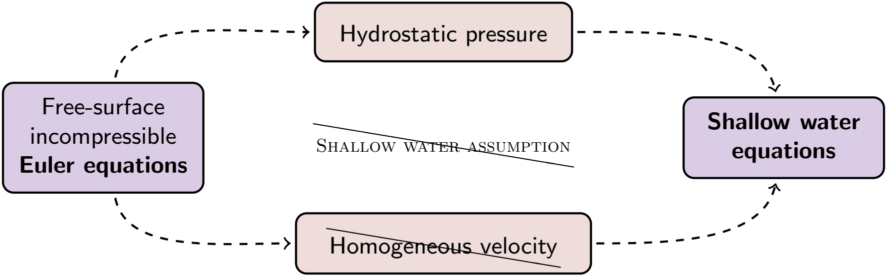
Literature about free-surface flows



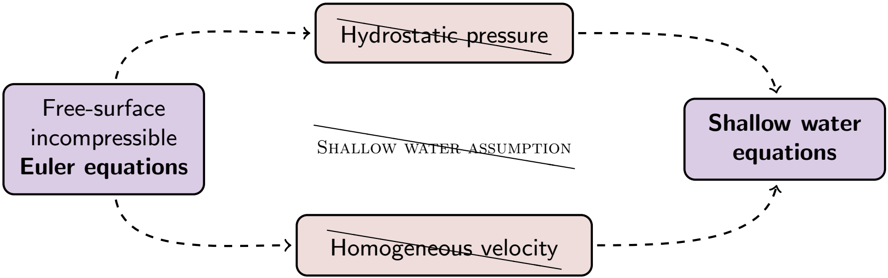
E. Audusse, M.-O. Bristeau, B. Perthame, J. Sainte-Marie, *A multilayer Saint-Venant system with mass exchanges for Shallow Water flows. Derivation and numerical validation* (**Math. Model. Numer. Anal.** 45(1), 2011)

F. Bouchut, V. Zeitlin, *A robust well-balanced scheme for multi-layer shallow water equations* (**Discrete Contin. Dyn. Syst. Ser. B** 13(4), 2010)

E.D. Fernández-Nieto, E.H. Koné, T. Morales de Luna, R. Bürger, *A multilayer shallow water system for polydisperse sedimentation* (**J. Comput. Phys.** 238, 2013)

Castro *et al.* '01 '04 '10, Narbona *et al.* '09 '13, . . .

Literature about free-surface flows



Derivation of multilayer non-hydrostatic models

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Y. Bai, K.F. Cheung, *Dispersion and nonlinearity of multi-layer non-hydrostatic free-surface flow* (**J. Fluid Mech.** 726, 2013)

Euler equations

Model

$$\begin{cases} \partial_x u + \partial_z w = 0 \\ \partial_t u + \partial_x(u^2 + p) + \partial_z(uw) = 0 \\ \partial_t w + \partial_x(uw) + \partial_z(w^2 + p) = -g \end{cases}$$

set in the domain $\Omega(t) = \left\{(x,z) \in \mathbb{R}^2 \;\middle|\; z_b(x) \leq z \leq \eta(t,x)\right\}$

Boundary conditions

$$\partial_t \eta(t,x) + u\big(t,x,\eta(t,x)\big)\partial_x \eta(t,x) - w\big(t,x,\eta(t,x)\big) = 0$$
$$p\big(t,x,\eta(t,x)\big) = p^{atm}(t,x)$$
$$u\big(t,x,z_b(x)\big)z_b'(x) - w\big(t,x,z_b(x)\big) = 0$$

together with well-prepared initial conditions

Pressure fields $p(t,x,z) = p^{atm}(t,x) + g\big(\eta(t,x) - z\big) + q(t,x,z)$

Euler equations

Model

$$\begin{cases} \partial_x u + \partial_z w = 0 \\ \partial_t u + \partial_x(u^2 + p) + \partial_z(uw) = 0 \\ \partial_t w + \partial_x(uw) + \partial_z(w^2 + p) = -g \end{cases}$$

set in the domain $\Omega(t) = \left\{(x,z) \in \mathbb{R}^2 \;\middle|\; z_b(x) \leq z \leq \eta(t,x)\right\}$

Boundary conditions

$$\partial_t \eta(t,x) + u\big(t,x,\eta(t,x)\big)\partial_x \eta(t,x) - w\big(t,x,\eta(t,x)\big) = 0$$

$$p\big(t,x,\eta(t,x)\big) = p^{atm}(t,x)$$

$$u\big(t,x,z_b(x)\big)z_b'(x) - w\big(t,x,z_b(x)\big) = 0$$

together with well-prepared initial conditions

Pressure fields $p(t,x,z) = p^{atm}(t,x) + g\big(\eta(t,x) - z\big) + \cancel{q(t,x,z)}$

Euler equations

Model

$$\begin{cases} \partial_x u + \partial_z w = 0 \\ \partial_t u + \partial_x(u^2 + q) + \partial_z(uw) = -\partial_x(g\eta + p^{atm}) \\ \partial_t w + \partial_x(uw) + \partial_z(w^2 + q) = 0 \end{cases}$$

set in the domain $\Omega(t) = \left\{(x,z) \in \mathbb{R}^2 \;\middle|\; z_b(x) \le z \le \eta(t,x)\right\}$

Boundary conditions

$$\partial_t \eta(t,x) + u\big(t,x,\eta(t,x)\big)\partial_x \eta(t,x) - w\big(t,x,\eta(t,x)\big) = 0$$

$$q\big(t,x,\eta(t,x)\big) = 0$$

$$u\big(t,x,z_b(x)\big)z_b'(x) - w\big(t,x,z_b(x)\big) = 0$$

together with well-prepared initial conditions

Euler equations

Model

$$\begin{cases} \partial_x u + \partial_z w = 0 \\ \partial_t u + \partial_x(u^2 + q) + \partial_z(uw) = -\partial_x(g\eta + p^{atm}) \\ \partial_t w + \partial_x(uw) + \partial_z(w^2 + q) = 0 \end{cases}$$

set in the domain $\Omega(t) = \left\{(x,z) \in \mathbb{R}^2 \;\middle|\; z_b(x) \leq z \leq \eta(t,x)\right\}$

Energy equality

$$\partial_t \left(\frac{u^2 + w^2}{2}\right) + \partial_x \left[u\left(\frac{u^2+w^2}{2} + q + g\eta\right)\right] + \partial_z \left[w\left(\frac{u^2+w^2}{2} + q + g\eta\right)\right] = 0$$

α – Depth-averaged Euler models

Model

$$\begin{cases} \partial_t h + \partial_x (h\bar{u}) = 0 \\ \partial_t (h\bar{u}) + \partial_x \left(h\bar{u}^2 + g\dfrac{h^2}{2} + hq \right) = -\left(\dfrac{\alpha^2}{2} q + gh \right) \partial_x z_b \\ \partial_t (h\bar{w}) + \partial_x (h\bar{u}\bar{w}) = \alpha q \\ \alpha\bar{w} + h\partial_x \bar{u} - \dfrac{\alpha^2}{2} \bar{u}\partial_x z_b = 0 \end{cases}$$

Literature

- $\alpha = 2$: see Aïssiouène, Bristeau, Godlewski, Sainte-Marie
- $\alpha = \sqrt{3}$ with a flat topography: see Lannes, ...

α – Depth-averaged Euler models

Model

$$\begin{cases} \partial_t h + \partial_x(h\bar{u}) = 0 \\ \partial_t(h\bar{u}) + \partial_x\left(h\bar{u}^2 + g\frac{h^2}{2} + hq\right) = -\left(\frac{\alpha^2}{2}q + gh\right)\partial_x z_b \\ \partial_t(h\bar{w}) + \partial_x(h\bar{u}\bar{w}) = \alpha q \\ \alpha\bar{w} + h\partial_x\bar{u} - \frac{\alpha^2}{2}\bar{u}\partial_x z_b = 0 \end{cases}$$

Energy equality

$$\partial_t\left(h\frac{\bar{u}^2 + \bar{w}^2}{2} + g\frac{h^2}{2} + ghz_b\right) + \partial_x\left[h\bar{u}\left(\frac{\bar{u}^2 + \bar{w}^2}{2} + q + g\eta\right)\right] = 0$$

Serre – Green-Naghdi model

Model

$$
\begin{cases}
\partial_t h + \partial_x (h\bar{u}) = 0 \\
\partial_t (h\bar{u}) + \partial_x \left(h\bar{u}^2 + g\dfrac{h^2}{2} + hq \right) = -\left(q_b + gh\right)\partial_x z_b \\
\partial_t (h\bar{w}) + \partial_x (h\bar{u}\bar{w}) = q_b \\
\partial_t (h\sigma) + \partial_x (h\sigma\bar{u}) = 2\sqrt{3}\left(q - \dfrac{q_b}{2} \right) \\
2\sqrt{3}\sigma + h\partial_x \bar{u} = 0 \\
\bar{w} - \sqrt{3}\sigma - \bar{u}\partial_x z_b = 0
\end{cases}
$$

Serre – Green-Naghdi model

Model

$$\begin{cases} \partial_t h + \partial_x (h\bar{u}) = 0 \\ \partial_t (h\bar{u}) + \partial_x \left(h\bar{u}^2 + g\dfrac{h^2}{2} + hq \right) = -\left(q_b + gh\right)\partial_x z_b \\ \partial_t (h\bar{w}) + \partial_x (h\bar{u}\bar{w}) = q_b \\ \partial_t (h\sigma) + \partial_x (h\sigma\bar{u}) = 2\sqrt{3}\left(q - \dfrac{q_b}{2}\right) \\ 2\sqrt{3}\sigma + h\partial_x \bar{u} = 0 \\ \bar{w} - \sqrt{3}\sigma - \bar{u}\partial_x z_b = 0 \end{cases}$$

Energy equality

$$\partial_t \left(h\frac{\bar{u}^2 + \bar{w}^2 + \sigma^2}{2} + g\frac{h^2}{2} + ghz_b \right) + \partial_x \left[h\bar{u} \left(\frac{\bar{u}^2 + \bar{w}^2 + \sigma^2}{2} + q + g\eta \right) \right] = 0$$

c

Serre – Green-Naghdi model

Equivalent formulation

$$\begin{cases} \partial_t h + \partial_x(hu) = 0 \\ \big(\mathcal{I}_d + \mathcal{T}[h, z_b]\big)(\partial_t u + u\partial_x u) + g\partial_x(h + z_b) + \mathcal{Q}[h, z_b]u = 0 \end{cases}$$

where

$$\begin{aligned} \mathcal{T}[h, z_b]v &= \mathcal{R}_1[h, z_b](\partial_x v) + \mathcal{R}_2[h, z_b](v\partial_x z_b) \\ \mathcal{Q}[h, z_b]v &= -2\mathcal{R}_1[h, z_b]\left((\partial_x v)^2\right) + \mathcal{R}_2[h, z_b](v^2\partial_{xx}^2 z_b) \end{aligned}$$

$$\begin{aligned} \mathcal{R}_1[h, z_b]w &= -\frac{1}{3h}\partial_x(h^3 w) - \frac{h}{2}w\partial_x z_b \\ \mathcal{R}_2[h, z_b]w &= \frac{1}{2h}\partial_x(h^2 w) + w\partial_x z_b \end{aligned}$$

Serre – Green-Naghdi model

Another formulation

$$\begin{cases} 12\dfrac{q}{h^*} - h^*\partial_x\left(\dfrac{\partial_x(h^*q)}{h^*}\right) - 6\dfrac{q_b}{h^*} - h^*\partial_x\left(\dfrac{q_b}{h^*}\partial_x z_b\right) \\ \qquad = 2h(\partial_x u)^2 + h\partial_x(g\partial_x(z_b+h) + \partial_x p^{atm}) \\ \left(4 + (\partial_x z_b)^2\right)\dfrac{q_b}{h^*} - 6\dfrac{q}{h^*} + \partial_x z_b \dfrac{\partial_x(h^*q)}{h^*} \\ \qquad = u^2\partial^2_{xx} z_b - (g\partial_x(z_b+h) + \partial_x p^{atm})\partial_x z_b \end{cases}$$

Energy

$$\int_I \frac{\left(\partial_x(h^*q) + q_b\partial_x z_b\right)^2 + 3q^2 + (3q-2q_b)^2}{h^*}\mathrm{d}x = \ell(q, q_b)$$

Serre – Green-Naghdi model

Compact form

$$\begin{cases} \partial_t h + \partial_x(h\bar{u}) = 0 \\ \partial_t(h\mathbf{X}) + \partial_x(h\bar{u}\mathbf{X}) + \nabla_{sgn}\mathbf{Q} - \mathbf{S}(h) = 0 \\ \nabla_{sgn}\cdot\mathbf{X} = 0 \end{cases}$$

with

$$\mathbf{X} = \begin{pmatrix} u \\ w \\ \sigma \end{pmatrix} \qquad \nabla_{sgn}\mathbf{Q} = \begin{pmatrix} \partial_x(hq) + q_b\partial_x z_b \\ -q_b \\ -2\sqrt{3}\left(q - \frac{q_b}{2}\right) \end{pmatrix}$$

$$\mathbf{Q} = \begin{pmatrix} q \\ q_b \end{pmatrix} \qquad \nabla_{sgn}\cdot\mathbf{X} = \begin{pmatrix} 2\sqrt{3}\sigma + h\partial_x\bar{u} \\ \bar{w} - \sqrt{3}\sigma - \bar{u}\partial_x z_b \end{pmatrix}$$

$$\mathbf{X}\cdot\nabla_{sgn}\mathbf{Q} = \partial_x(hqu) - \mathbf{Q}\cdot\nabla_{sgn}\cdot\mathbf{X}$$

Numerical strategy

Model

$$\begin{cases} \partial_t h + \partial_x(h\bar{u}) = 0 \\ \partial_t(h\mathbf{X}) + \partial_x(h\bar{u}\mathbf{X}) + \nabla_{sgn}\mathbf{Q} - \mathbf{S}(h) = 0 \\ \nabla_{sgn}\cdot\mathbf{X} = 0 \end{cases}$$

Semi-discrete form

$$\begin{cases} \dfrac{h^{n+1/2} - h^n}{\Delta t} + \partial_x(h^n u^n) = 0, \\ \dfrac{(h\mathbf{X})^{n+1/2} - (h\mathbf{X})^n}{\Delta t} + \partial_x\left(h^n u^n \mathbf{X}^n\right) = \mathbf{S}(h^n), \end{cases}$$

Numerical strategy

Model

$$\begin{cases} \partial_t h + \partial_x (h\bar{u}) = 0 \\ \partial_t (h\mathbf{X}) + \partial_x (h\bar{u}\mathbf{X}) + \nabla_{sgn} \mathbf{Q} - \mathbf{S}(h) = 0 \\ \nabla_{sgn} \cdot \mathbf{X} = 0 \end{cases}$$

Semi-discrete form

$$\begin{cases} \dfrac{h^{n+1} - h^{n+1/2}}{\Delta t} = 0, \\ \dfrac{(h\mathbf{X})^{n+1} - (h\mathbf{X})^{n+1/2}}{\Delta t} + \nabla_{sgn} \mathbf{Q}^{n+1} = 0, \\ \nabla_{sgn} \cdot \mathbf{X}^{n+1} = \mathbf{0}. \end{cases}$$

Choices

Mixed formulation

$$\begin{cases} \mathbf{X}^{n+1} + \dfrac{\Delta t}{h^{n+1/2}} \nabla_{sgn} \mathbf{Q}^{n+1} = \mathbf{X}^{n+1/2}, \\ \nabla_{sgn} \cdot \mathbf{X}^{n+1} = \mathbf{0}. \end{cases}$$

Projection-correction formulation

$$\begin{cases} -\nabla_{sgn} \cdot \left(\dfrac{1}{h^{n+1/2}} \nabla_{sgn} \mathbf{Q}^{n+1} \right) = -\dfrac{1}{\Delta t} \nabla_{sgn} \cdot \mathbf{X}^{n+1/2}, \\ \mathbf{X}^{n+1} = \mathbf{X}^{n+1/2} - \dfrac{\Delta t}{h^{n+1/2}} \nabla_{sgn} \mathbf{Q}^{n+1}. \end{cases}$$

Fully discrete scheme #1

Mixed formulation

$$\begin{cases} \dfrac{h^{n+1/2}}{\Delta t}\mathbf{X}^{n+1} + \nabla_{sgn}\,\mathbf{Q}^{n+1} = \dfrac{h^{n+1/2}}{\Delta t}\mathbf{X}^{n+1/2}, \\ \nabla_{sgn}\cdot\,\mathbf{X}^{n+1} = \mathbf{0}. \end{cases}$$

Colocated scheme

$$\frac{1}{\Delta t}\overline{\mathcal{H}}\mathbf{X}^{n+1} + B\mathbf{Q}^{n+1} = \frac{1}{\Delta t}\overline{\mathcal{H}}\mathbf{X}^{n+1/2} - \widehat{\mathbf{0}}$$

- $\overline{\mathcal{H}} \in \mathcal{M}_{3N,3N}(\mathbb{R})$ is block-diagonal with block entries $\mathcal{H}_i = h_i^{n+1/2}$
- $B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline \mathbf{0} & -\mathcal{I}_N \\ \hline -2\sqrt{3}\mathcal{I}_N & \sqrt{3}\mathcal{I}_N \end{array}\right) \in \mathcal{M}_{3N,2N}(\mathbb{R})$ one-to-one

Fully discrete scheme #1

Mixed formulation

$$\begin{cases} \dfrac{h^{n+1/2}}{\Delta t}\mathbf{X}^{n+1} + \nabla_{sgn}\,\mathbf{Q}^{n+1} = \dfrac{h^{n+1/2}}{\Delta t}\mathbf{X}^{n+1/2}, \\ \nabla_{sgn}\cdot\,\mathbf{X}^{n+1} = \mathbf{0}. \end{cases}$$

Colocated scheme

$$\left(\begin{array}{c|c} \overline{\mathcal{H}}/\Delta t & B \\ \hline B^T & \mathbf{0} \end{array}\right) \left(\begin{array}{c} \mathbf{X}^{n+1} \\ \hline \mathbf{Q}^{n+1} \end{array}\right) = \left(\begin{array}{c} \overline{\mathcal{H}}\mathbf{X}^{n+1/2}/\Delta t - \widehat{\mathbf{0}} \\ \hline \tilde{\mathbf{0}} \end{array}\right).$$

Resolution by means of the Uzawa method.

Fully discrete scheme #2

Projection-correction formulation

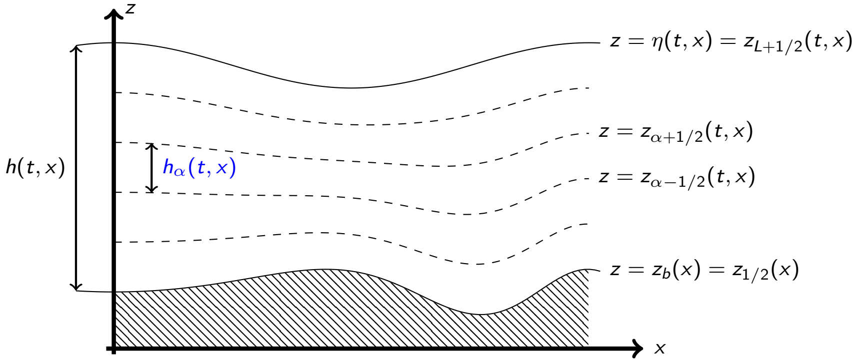
$$\begin{cases} -\nabla_{sgn}\cdot\left(\dfrac{1}{h^{n+1/2}}\nabla_{sgn}\mathbf{Q}^{n+1}\right) = -\dfrac{1}{\Delta t}\nabla_{sgn}\cdot\mathbf{X}^{n+1/2}, \\ \mathbf{X}^{n+1} = \mathbf{X}^{n+1/2} - \dfrac{\Delta t}{h^{n+1/2}}\nabla_{sgn}\mathbf{Q}^{n+1}. \end{cases}$$

Colocated scheme

- Strategy #1: direct discretisation
- Strategy #2: inferred from the mixed formulation

$$B^T\overline{\mathcal{H}}^{-1}B\mathbf{Q}^{n+1} = \frac{1}{\Delta t}B^T\mathbf{X}^{n+1/2} - B^T\overline{\mathcal{H}}^{-1}\widehat{\mathbf{0}} - \frac{1}{\Delta t}\widetilde{\mathbf{0}}$$

Multilayer framework



Height decomposition: $h_\alpha(t,x) = \frac{h(t,x)}{L}$

Non-hydrostatic multilayer model

$$\partial_t h + \partial_x (h\bar{u}) = 0 \qquad \bar{u} = \sum_{\alpha=1}^{L} \ell_\alpha \bar{u}_\alpha$$

and for $\alpha \in \{1, \dots, L\}$

$$\partial_t(h_\alpha \bar{u}_\alpha) + \partial_x \left(h_\alpha \bar{u}_\alpha^2 + h_\alpha \bar{q}_\alpha\right) + \tilde{u}_{\alpha+1/2}\Upsilon_{\alpha+1/2} - \partial_x z_{\alpha+1/2} q_{\alpha+1/2} - \tilde{u}_{\alpha-1/2}\Upsilon_{\alpha-1/2} + \partial_x z_{\alpha-1/2} q_{\alpha-1/2} = -g h_\alpha \partial_x \eta$$

$$\partial_t(h_\alpha \bar{w}_\alpha) + \partial_x (h_\alpha \bar{u}_\alpha \bar{w}_\alpha) + \tilde{w}_{\alpha+1/2}\Upsilon_{\alpha+1/2} + q_{\alpha+1/2} - \tilde{w}_{\alpha-1/2}\Upsilon_{\alpha-1/2} - q_{\alpha-1/2} = 0$$

$$\partial_t(h_\alpha \sigma_\alpha) + \partial_x(h_\alpha \sigma_\alpha \bar{u}_\alpha) = 2\sqrt{3}\left[\bar{q}_\alpha - \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2} - \Upsilon_{\alpha+1/2}\left(\frac{h_\alpha \partial_x \bar{u}_\alpha}{12} + \frac{\tilde{w}_{\alpha+1/2} - \bar{w}_\alpha}{2}\right) + \Upsilon_{\alpha-1/2}\left(\frac{h_\alpha \partial_x \bar{u}_\alpha}{12} + \frac{\bar{w}_\alpha - \tilde{w}_{\alpha-1/2}}{2}\right)\right]$$

together with diagnostic equations

$$2\sqrt{3}\sigma_\alpha + h_\alpha \partial_x \bar{u}_\alpha = 0, \qquad \bar{w}_{\alpha+1} - \bar{w}_\alpha - (\bar{u}_{\alpha+1} - \bar{u}_\alpha)\partial_x z_{\alpha+1/2} - \sqrt{3}(\sigma_{\alpha+1} + \sigma_\alpha) = 0$$

$$w_1 - u_1 \partial_x z_b - \sqrt{3}\sigma_1 = 0 \qquad q_{L+1/2} = 0$$

Non-hydrostatic multilayer model

$$\partial_t h + \partial_x (h\bar{u}) = 0, \qquad \bar{u} = \sum_{\alpha=1}^{L} \ell_\alpha \bar{u}_\alpha,$$

and for $\alpha \in \{1, \ldots, L\}$

$$\partial_t(h_\alpha \mathbf{X}_\alpha) + \partial_x(h_\alpha \bar{u}_\alpha \mathbf{X}_\alpha) + \Upsilon_{\alpha+1/2}\widetilde{\mathbf{X}}_{\alpha+1/2} - \Upsilon_{\alpha-1/2}\widetilde{\mathbf{X}}_{\alpha-1/2} + \nabla^\alpha_{nhml}\,\mathbf{Q}_\alpha - \mathbf{S}_\alpha(h) = 0$$

$$\nabla^\alpha_{nhml}\cdot \mathbf{X}_\alpha = 0$$

where $\widetilde{\mathbf{X}}_{\alpha+1/2} = \gamma_{\alpha+1/2}\mathbf{X}^-_{\alpha+1/2} + (1-\gamma_{\alpha+1/2})\mathbf{X}^+_{\alpha+1/2}$.

Energy (in)equality Provided that $\gamma_{\alpha+1/2} = \frac{1}{2}[1 + \lambda\,\mathrm{sign}(\Upsilon_{\alpha+1/2})]$ for $\lambda \geq 0$:

$$\partial_t \left[\sum_{\alpha=1}^{L} h_\alpha \left(\frac{\bar{u}_\alpha^2 + \bar{w}_\alpha^2 + \sigma_\alpha^2}{2} + g z_\alpha\right)\right] + \partial_x \left[\sum_{\alpha=1}^{L} h_\alpha \bar{u}_\alpha \left(\frac{\bar{u}_\alpha^2 + \bar{w}_\alpha^2 + \sigma_\alpha^2}{2} + g\eta + q_\alpha\right)\right] \leq 0$$

Conclusion

Achievements

- Comparisons between different formulations of the same model
- Derivation of a class of multilayer non-hydrostatic models as semi-discretisations of the Euler equations
- Analysis of physical properties (energy, hydrodynamic balances, dispersive effects)

On-going works

- Boundary conditions
- Efficency of the numerical methods: extensions to 3D and multilayer cases (CEMRACS 2019)

Thank you for your attention