Numerical schemes for Euler equations: a physically admissible extension to order 2

Yohan Penel[1]

[1]Team ANGE (CEREMA, Inria, UPMC, CNRS)

Joint work with
C. Calgaro & E. Creusé (Univ. Lille 1, Inria Lille)
T. Goudon (Inria Sophia-Antipolis)

Seminario IMUS

Sevilla – February, 24th. 2016

Issue

- **System of conservation laws** modelling a physical phenomenon like in fluid mechanics, ...

$$\begin{cases} \partial_t \mathbf{W} + \nabla \cdot \mathcal{F}(\mathbf{W}) = 0, \\ \mathbf{W}(0, \mathbf{x}) = \mathbf{W}_0(\mathbf{x}). \end{cases}$$

- **Physical constraints**: $\mathbf{W} \in \mathcal{W}$
  - Maximum principle (Euler for incompressible fluids: density)
  - Positivity (Euler for compressible fluids: density and pressure)
- These constraints must be satisfied:
  - at the **continuous** level (relevance of the mathematical model)
  - at the **discrete** level (robustness of the numerical scheme)

Example

Euler equations for a 2D perfect fluid:

$$\mathbf{W} = {}^t(\rho, \rho\mathbf{u}, \rho E), \quad \mathcal{F} = {}^t(\rho\mathbf{u}, \rho\mathbf{u}\otimes\mathbf{u} + p\mathcal{I}_2, (\rho E + p)\mathbf{u})$$

$$\mathcal{W} = \left\{ \mathbf{W} \in \mathbb{R}^4 \;:\; \rho = W_1 > 0 \;\text{ et }\; p = (\gamma - 1)\left[W_4 - \frac{W_2^2 + W_3^2}{2W_1}\right] > 0 \right\}$$

Riemann problem: $\mathbf{W}_0(\mathbf{x}) = \begin{cases} \mathbf{W}_l, & \text{if } x_1 < 0, \\ \mathbf{W}_r, & \text{if } x_1 > 0. \end{cases}$

Rarefaction waves and vacuum (Einfeldt, Munz, Roe & Sjögreen, 1991):

- $\rho_0 > 0,\ u_0 > 0,\ E_0 > u_0^2/2$
- $\mathbf{W}_l = (\rho_0, -\rho_0 u_0, 0, \rho_0 E_0)$ and $\mathbf{W}_r = (\rho_0, \rho_0 u_0, 0, \rho_0 E_0)$
- If $\dfrac{4\gamma}{3\gamma - 1} E_0 > u_0^2$, then density and pressure remain **positive**.

Positivity-preserving schemes

1st order

- Einfeldt *et al.* (1991), Bouchut (2004)
- **Godunov, Rusanov, HLLs** / **Roe**

2nd order: FV schemes + MUSCL strategy

- **Scalar equations**: Clain & Clauzon (2010), Calgaro *et al.* (2010)
  - Modification of limiters
  - Adaptation of the CFL condition
- **Systems of conservation laws**:
  - . . . *monoslope* reconstruction: Perthame & Shu (1996)
  - . . . *multislope* reconstruction: Berthon (2006)
    - Method based on the **convexity** of $\mathcal{W}$
    - Adaptation of the **reconstruction procedure**
    - Addition of a **fictitious state** in each cell
    - Reformulation of the 2nd order 2D scheme as a **convex combination** of 1st order 1D schemes

Positivity-preserving schemes

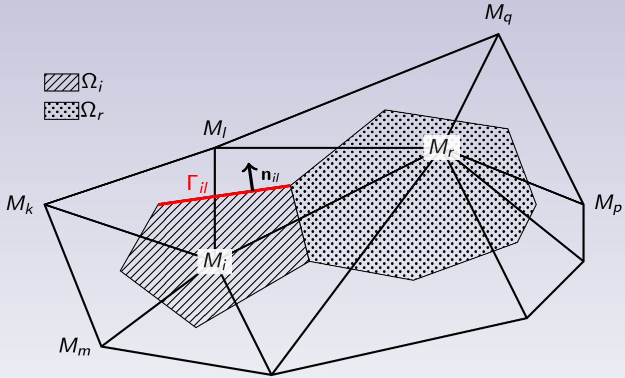
1st order

- Einfeldt *et al.* (1991), Bouchut (2004)
- **Godunov, Rusanov, HLLs** / **Roe**

2nd order: FV schemes + MUSCL strategy

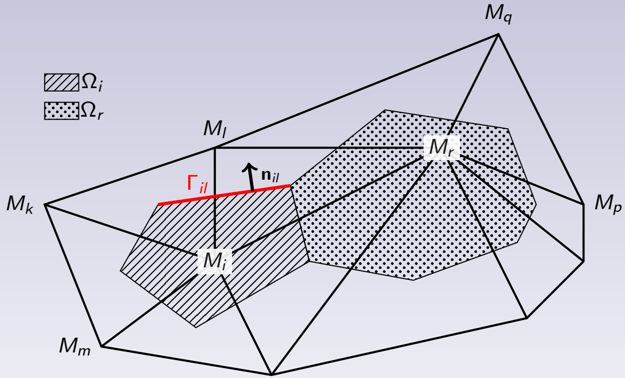
- **Scalar equations**: Clain & Clauzon (2010), Calgaro *et al.* (2010)
  - Modification of limiters
  - Adaptation of the CFL condition
- **Systems of conservation laws**:
  - . . . *monoslope* reconstruction: Perthame & Shu (1996)
  - . . . *multislope* reconstruction: Berthon (2006)
    - Method based on the **convexity** of $\mathcal{W}$
    - Adaptation of the **reconstruction procedure**
    - Addition of a **fictitious state** in each cell
    - Reformulation of the 2nd order 2D scheme as a **convex combination** of 1st order 1D schemes

Finite volume schemes: 1st-order



$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \Delta t^n \sum_{j \in \mathcal{V}(i)} \frac{|\Gamma_{ij}|}{|\Omega_i|} \mathcal{F}\left(\mathbf{W}_i^n, \mathbf{W}_j^n, \mathbf{n}_{ij}\right)$$

Finite volume schemes: 2nd-order (MUSCL)



$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \Delta t^n \sum_{j \in \mathcal{V}(i)} \frac{|\Gamma_{ij}|}{|\Omega_i|} \mathcal{F}\left(\mathbf{W}_{ij}^n, \mathbf{W}_{ji}^n, \mathbf{n}_{ij}\right)$$

Modification of Berthon's strategy (2006)

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \Delta t^n \sum_{j\in\mathcal{V}(i)} \frac{|\Gamma_{ij}|}{|\Omega_i|}\mathcal{F}\left(\mathbf{W}_{ij}^n, \mathbf{W}_{ji}^n, \mathbf{n}_{ij}\right)$$

Modification of Berthon's strategy (2006)

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \Delta t^n \sum_{j\in\mathcal{V}(i)} \frac{|\Gamma_{ij}|}{|\Omega_i|} \mathcal{F}\left(\mathbf{W}_{ij}^n, \mathbf{W}_{ji}^n, \mathbf{n}_{ij}\right)$$

$$\mathbf{W}_i^n = \frac{|\Omega_i^*|}{|\Omega_i|}\mathbf{W}_i^* + \sum_{j\in\mathcal{V}(i)} \frac{|\Omega_{ij}|}{|\Omega_i|}\mathbf{W}_{ij}^n$$

Modification of Berthon's strategy (2006)

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \Delta t^n \sum_{j\in\mathcal{V}(i)} \frac{|\Gamma_{ij}|}{|\Omega_i|}\mathcal{F}\left(\mathbf{W}_{ij}^n, \mathbf{W}_{ji}^n, \mathbf{n}_{ij}\right)$$

$$\mathbf{W}_i^n = \frac{|\Omega_i^*|}{|\Omega_i|}\mathbf{W}_i^* + \sum_{j\in\mathcal{V}(i)} \frac{|\Omega_{ij}|}{|\Omega_i|}\mathbf{W}_{ij}^n$$

$$\mathbf{W}_i^{n+1} = \frac{|\Omega_i^*|}{|\Omega_i|}\overline{\mathbf{W}}_i^* + \sum_{j\in\mathcal{V}(i)} \frac{|\Omega_{ij}|}{|\Omega_i|}\overline{\mathbf{W}}_{ij}$$

Modification of Berthon's strategy (2006)

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \Delta t^n \sum_{j \in \mathcal{V}(i)} \frac{|\Gamma_{ij}|}{|\Omega_i|} \mathcal{F}\left(\mathbf{W}_{ij}^n, \mathbf{W}_{ji}^n, \mathbf{n}_{ij}\right)$$

$$\mathbf{W}_i^n = \frac{|\Omega_i^*|}{|\Omega_i|}\mathbf{W}_i^* + \sum_{j \in \mathcal{V}(i)} \frac{|\Omega_{ij}|}{|\Omega_i|}\mathbf{W}_{ij}^n$$

$$\mathbf{W}_i^{n+1} = \frac{|\Omega_i^*|}{|\Omega_i|}\overline{\mathbf{W}}_i^* + \sum_{j \in \mathcal{V}(i)} \frac{|\Omega_{ij}|}{|\Omega_i|}\overline{\mathbf{W}}_{ij}$$

$$\overline{\mathbf{W}} = \mathbf{W}^n - \frac{\Delta t}{\ell}\left[\mathcal{F}(\mathbf{W}^n, \mathbf{V}^n, \mathbf{n}) - \mathcal{F}(\mathbf{W}^n, \mathbf{W}^n, \mathbf{n})\right]$$

for a suitable 1D flux $\mathcal{F}$ and a small enough time step Δt

Modification of Berthon's strategy (2006)

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \Delta t^n \sum_{j\in\mathcal{V}(i)} \frac{|\Gamma_{ij}|}{|\Omega_i|} \mathcal{F}\left(\mathbf{W}_{ij}^n, \mathbf{W}_{ji}^n, \mathbf{n}_{ij}\right)$$

$$\mathbf{W}_i^n = \eta_i^* \mathbf{W}_i^* + (1-\eta_i^*) \sum_{j\in\mathcal{V}(i)} \eta_{ij} \mathbf{W}_{ij}^n$$

$$\mathbf{W}_i^{n+1} = \eta_i^* \overline{\mathbf{W}}_i^* + (1-\eta_i^*) \sum_{j\in\mathcal{V}(i)} \eta_{ij} \overline{\mathbf{W}}_{ij}$$

$$\overline{\mathbf{W}} = \mathbf{W}^n - \frac{\Delta t}{\ell}\left[\mathcal{F}(\mathbf{W}^n, \mathbf{V}^n, \mathbf{n}) - \mathcal{F}(\mathbf{W}^n, \mathbf{W}^n, \mathbf{n})\right]$$

for a suitable 1D flux $\mathcal{F}$ and a small enough time step Δt

Modification of Berthon's strategy (2006)

How to choose η_i^* and η_{ij}?

Modification of Berthon's strategy (2006)

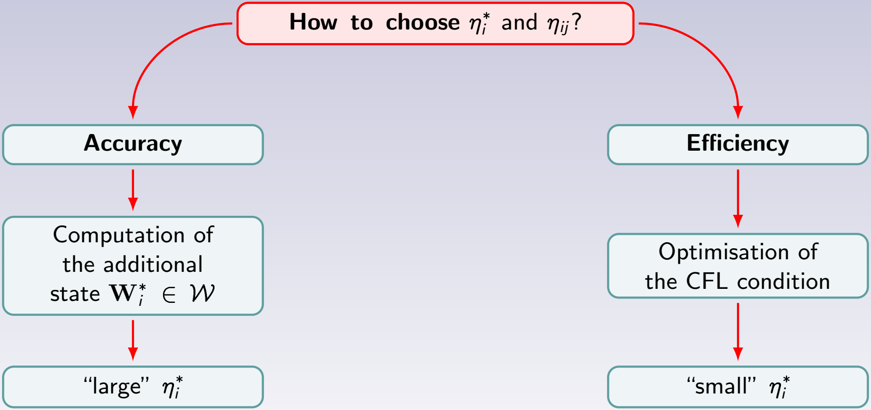
How to choose η_i^* and η_{ij}?

Accuracy

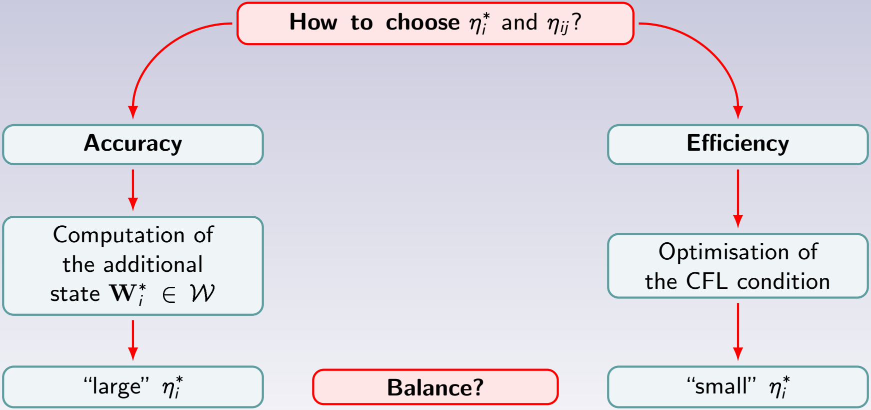
Computation of the additional state $\mathbf{W}_i^* \in \mathcal{W}$

"large" η_i^*

Modification of Berthon's strategy (2006)



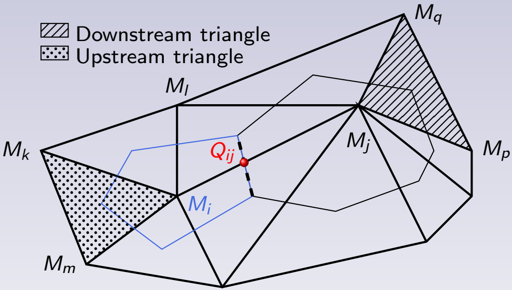
Modification of Berthon's strategy (2006)



Outline

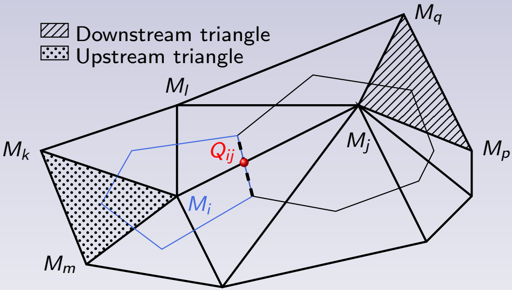
Reconstruction across interfaces

Issue: ensure that $\mathbf{W}_{ij} \in \mathcal{W}$



Reconstruction across interfaces

Issue: ensure that $\mathbf{W}_{ij} \in \mathcal{W}$

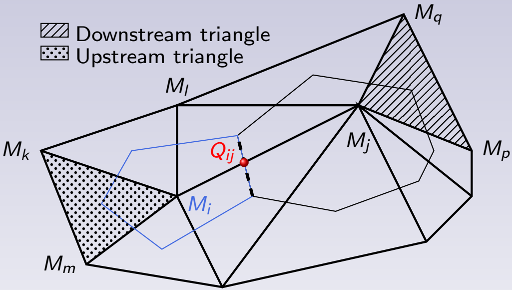


Reconstruction of physical variables $\mathbf{U} = (\rho, \mathbf{u}, p) = \kappa(\mathbf{W})$

$$\xi_{ij} = \xi_i + \alpha_{ij}\varphi(r_{ij})(\xi_j - \xi_i), \qquad r_{ij} = \frac{\overline{\Delta\xi}_{ij}^{up}}{\xi_j - \xi_i}$$

Reconstruction across interfaces

Issue: ensure that $\mathbf{W}_{ij} \in \mathcal{W}$

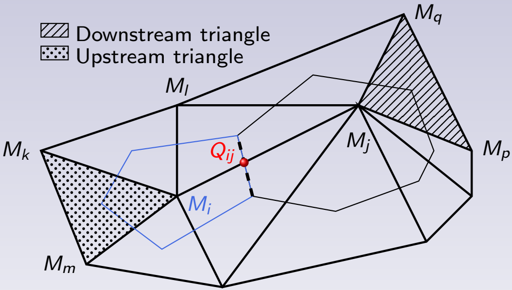


Use of τ-limiters

$$\alpha_{ij} = \frac{M_i Q_{ij}}{M_i M_j}, \quad \tau = \min_{i,j} \frac{1}{\alpha_{ij}} \in [1,2], \quad 0 \leqslant \varphi(r) \leqslant \min(\tau r, \tau)$$

Reconstruction across interfaces

Issue: ensure that $\mathbf{W}_{ij} \in \mathcal{W}$



Gradient with conservative variables

$$\mathbf{W}_{ij} = \mathbf{W}_i + \Delta\mathbf{W}_{ij}, \qquad \Delta\mathbf{W}_{ij} = \kappa^{-1}\big(\mathbf{U}_i + \Delta\mathbf{U}_{ij}\big) - \mathbf{W}_i$$

Key-point of the procedure

The additional state must be physically admissible

$$\mathbf{W}_{ij}^n = \mathbf{W}_i^n + \Delta\mathbf{W}_{ij}^n,$$

$$\mathbf{W}_i^* = \frac{1}{\eta_i^*}\left[\mathbf{W}_i^n - (1-\eta_i^*)\sum_{j\in\mathcal{V}(i)}\eta_{ij}\mathbf{W}_{ij}^n\right] = \mathbf{W}_i^n - \frac{1-\eta_i^*}{\eta_i^*}\sum_{j\in\mathcal{V}(i)}\eta_{ij}\Delta\mathbf{W}_{ij}^n$$

Key-point of the procedure

The additional state must be physically admissible

$$\mathbf{W}_{ij}^n = \mathbf{W}_i^n + \beta_i^n \Delta\mathbf{W}_{ij}^n,$$

$$\mathbf{W}_i^* = \frac{1}{\eta_i^*}\left[\mathbf{W}_i^n - (1-\eta_i^*)\beta_i^n \sum_{j\in\mathcal{V}(i)} \eta_{ij}\mathbf{W}_{ij}^n\right] = \mathbf{W}_i^n - \frac{1-\eta_i^*}{\eta_i^*}\beta_i^n \sum_{j\in\mathcal{V}(i)} \eta_{ij}\Delta\mathbf{W}_{ij}^n$$

$$\beta_i^n \in [0,1] \implies \mathbf{W}_{ij}^n = (1-\beta_i^n)\mathbf{W}_i^n + \beta_i^n\left[\mathbf{W}_i^n + \Delta\mathbf{W}_{ij}^n\right] \in \mathcal{W}$$

Key-point of the procedure

The additional state must be physically admissible

$$\mathbf{W}_{ij}^n = \mathbf{W}_i^n + \beta_i^n \Delta\mathbf{W}_{ij}^n,$$

$$\mathbf{W}_i^* = \frac{1}{\eta_i^*}\left[\mathbf{W}_i^n - (1-\eta_i^*)\beta_i^n \sum_{j\in\mathcal{V}(i)} \eta_{ij}\mathbf{W}_{ij}^n\right] = \mathbf{W}_i^n - \frac{1-\eta_i^*}{\eta_i^*}\beta_i^n \sum_{j\in\mathcal{V}(i)} \eta_{ij}\Delta\mathbf{W}_{ij}^n$$

$$\beta_i^n \in [0,1] \implies \mathbf{W}_{ij}^n = (1-\beta_i^n)\mathbf{W}_i^n + \beta_i^n\left[\mathbf{W}_i^n + \Delta\mathbf{W}_{ij}^n\right] \in \mathcal{W}$$

Let us set

$$\Delta\mathbf{W}_i^* = -\sum_{j\in\mathcal{V}(i)} \eta_{ij}\Delta\mathbf{W}_{ij}^n$$

so that $\mathbf{W}_i^* = \mathbf{W}_i^n + \dfrac{1-\eta_i^*}{\eta_i^*}\beta_i^n\Delta\mathbf{W}_i^*$.

Choice of β_i^n prescribed by $\rho_i^* > 0$ and $p_i^* > 0$

Choice of β_i^n

$$\rho_i^* > 0 \iff \mathcal{P}_1\left(\frac{1-\eta_i^*}{\eta_i^*}\beta_i^n\right) := 1 + D_i^n\left[\frac{1-\eta_i^*}{\eta_i^*}\beta_i^n\right] > 0$$

$$p_i^* > 0 \iff \mathcal{P}_2\left(\frac{1-\eta_i^*}{\eta_i^*}\beta_i^n\right) := 1 + B_i^n\left[\frac{1-\eta_i^*}{\eta_i^*}\beta_i^n\right] + A_i^n\left[\frac{1-\eta_i^*}{\eta_i^*}\beta_i^n\right]^2 > 0$$

Density Given $D_i^n = \dfrac{\Delta\rho_i^*}{\rho_i^n}$, we derive a first condition:

$$\beta_i^n \leqslant \beta_i^{(\rho)} := \min\left\{1, \frac{\eta_i^*}{1-\eta_i^*}\gamma_i^{(\rho)}\right\} \quad \text{with} \quad \gamma_i^{(\rho)} = \begin{cases} +\infty, & \text{if } D_i^n \geq 0, \\ -\dfrac{1}{D_i^n}, & \text{otherwise.} \end{cases}$$

Pressure $\beta_i^n \leqslant \beta_i^{(p)}$ with ...

Convex combinations

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \Delta t^n \sum_{j\in\mathcal{V}(i)} \frac{|\Gamma_{ij}|}{|\Omega_i|}\boldsymbol{\mathcal{F}}\left(\mathbf{W}_{ij}^n, \mathbf{W}_{ji}^n, \mathbf{n}_{ij}\right)$$

$$\mathbf{W}_i^n = \eta_i^*\mathbf{W}_i^* + (1-\eta_i^*)\sum_{j\in\mathcal{V}(i)} \eta_{ij}\mathbf{W}_{ij}^n$$

$$\mathbf{W}_i^{n+1} = \eta_i^*\overline{\mathbf{W}}_i^* + (1-\eta_i^*)\sum_{j\in\mathcal{V}(i)} \eta_{ij}\overline{\mathbf{W}}_{ij}$$

Convex combinations

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \Delta t^n \sum_{j\in\mathcal{V}(i)} \frac{|\Gamma_{ij}|}{|\Omega_i|} \boldsymbol{\mathcal{F}}\left(\mathbf{W}_{ij}^n, \mathbf{W}_{ji}^n, \mathbf{n}_{ij}\right)$$

$$\mathbf{W}_i^n = \eta_i^* \mathbf{W}_i^* + (1-\eta_i^*) \sum_{j\in\mathcal{V}(i)} \eta_{ij} \mathbf{W}_{ij}^n$$

$$\mathbf{W}_i^{n+1} = \eta_i^* \overline{\mathbf{W}}_i^* + (1-\eta_i^*) \sum_{j\in\mathcal{V}(i)} \eta_{ij} \overline{\mathbf{W}}_{ij}$$

$$\overline{\mathbf{W}}_{ij} = \mathbf{W}_{ij}^n - \Delta t^n \sum_{k=1}^{4} \zeta_{ij,k} \boldsymbol{\mathcal{F}}\left(\mathbf{W}_{ij}^n, \mathbf{W}_{ij,k}^n, \mathbf{n}_{ij,k}\right), \quad j \in \mathcal{V}(i)$$

$$\begin{cases} \overline{\mathbf{W}}_{ij} = \displaystyle\sum_{k=1}^{4} \frac{\zeta_{ij,k}}{\mu_{ij,k}} \overline{\mathbf{W}}_{ij,k} \\ \overline{\mathbf{W}}_{ij,k} = \mathbf{W}_{ij}^n - \Delta t^n \mu_{ij,k} \left[\boldsymbol{\mathcal{F}}(\mathbf{W}_{ij}^n, \mathbf{W}_{ij,k}^n, \mathbf{n}_{ij,k}) - \boldsymbol{\mathcal{F}}(\mathbf{W}_{ij}^n, \mathbf{W}_{ij}^n, \mathbf{n}_{ij,k})\right] \end{cases}$$

Assumptions

Flux In addition to classical properties, we assume

$$\forall\,(V,W)\in\mathcal{W}^2,\ W-\frac{\Delta t}{\ell}[\mathcal{F}(W,V)-\mathcal{F}(W,W)]\in\mathcal{W},$$

under the CFL condition $\Delta t\max_k|\lambda_k(V,W)|\leqslant\alpha_0\ell.$

CFL Condition

$$\Delta t^n\times\max_{j\in\mathcal{V}(i)}\left\{\mu_{ij}^*,\max_{1\le k\le 4}\mu_{ij,k}\right\}\times\bar{\lambda}_i^n\le\alpha_0$$

$$\bar{\lambda}_i^n:=\max_{\substack{j\in\mathcal{V}(i)\\1\le k\le 4}}\left\{|\mathbf{u}_{ij}^n\cdot\mathbf{n}_{ij,k}|+c_{ij}^n,|\mathbf{u}_i^*\cdot\mathbf{n}_{ij,k}|+c_i^*\right\}$$

Aim Choose η_i^* and (η_{ij}) such that Δt^n is maximal, *i.e.* such that $\bar{\lambda}_i^n$ and $\max\{\mu_{ij}^*,\mu_{ij,k}\}$ are minimal.

Constraints

Sign

- $\mu_{ij,k} \geq 0$, $\mu_{ij}^* \geq 0$
- $\zeta_{ij,k} \geq 0$

Consistency of the decomposition

- $\eta_{ij}\zeta_{ij,2} = \eta_{ik}\zeta_{ik,4}$
- $(1-\eta_i^*)\eta_{ij}\zeta_{ij,1} = \eta_i^*\zeta_{ij}^*$
- $(1-\eta_i^*)\eta_{ij}\zeta_{ij,3} = \frac{|\Gamma_{ij}|}{|\Omega_i|}$
- $\displaystyle\sum_{k=1}^{4} \frac{\zeta_{ij,k}}{\mu_{ij,k}} = 1$, $\displaystyle\sum_{k=1}^{4} \frac{\zeta_{ij}^*}{\mu_{ij}^*} = 1$
- $\displaystyle\sum_{k=1}^{4} \zeta_{ij,k}\mathbf{n}_{ij,k} = \mathbf{0}$

Resolution

With $X = \zeta_{ij_0,1} \leq X_{max} := \dfrac{|\Gamma_{ij_0}|}{(1-\eta_i^*)\eta_{ij_0}|\Omega_i|}$ for some $j_0 \in \mathcal{V}(i)$, we have

$$\zeta_{ij,1} = \frac{|\Gamma_{ij}|}{|\Gamma_{ij_0}|}\frac{\eta_{ij_0}}{\eta_{ij}}X, \qquad \zeta_{ij}^* = \frac{|\Gamma_{ij}|}{|\Gamma_{ij_0}|}\frac{(1-\eta_i^*)\eta_{ij_0}}{\eta_i^*}X, \qquad \zeta_{ij,3} = \frac{|\Gamma_{ij}|}{(1-\eta_i^*)\eta_{ij}|\Omega_i|}$$

$$\zeta_{ij,2} = \frac{M_i G_{ijl}}{\eta_{ij}}\left[\frac{1}{(1-\eta_i^*)|\Omega_i|} - \frac{\eta_{ij_0}}{|\Gamma_{ij_0}|}X\right], \quad \zeta_{ij,4} = \frac{M_i G_{ijk}}{\eta_{ij}}\left[\frac{1}{(1-\eta_i^*)|\Omega_i|} - \frac{\eta_{ij_0}}{|\Gamma_{ij_0}|}X\right]$$

$$\mu_{ij,k} = \frac{|\partial \mathrm{T}_{ij}|}{(1-\eta_i^*)\eta_{ij}|\Omega_i|} - \frac{X}{|\Gamma_{ij_0}|}\frac{\eta_{ij_0}}{\eta_{ij}}\left(|\partial \mathrm{T}_{ij}| - 2|\Gamma_{ij}|\right), \qquad \mu_{ij}^* = \frac{|\partial\Omega_i|}{|\Gamma_{ij_0}|}\frac{(1-\eta_i^*)\eta_{ij_0}}{\eta_i^*}X$$

$$\mu_i^{\mathrm{opt}} := \min_{0\leq X\leq X_{max}} \max_{j\in\mathcal{V}(i)} \left\{\mu_{ij}^*(X), \max_{1\leq k\leq 4}\mu_{ij,k}(X)\right\}.$$

Resolution

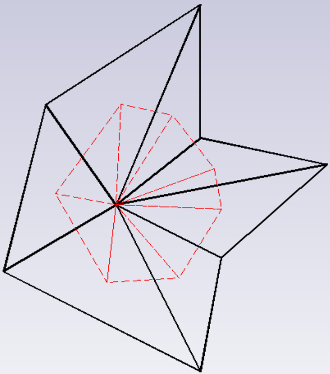
Optimisation The optimal coefficient reads

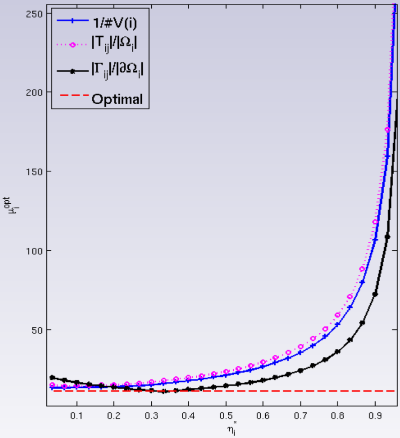
$$\mu_i^{\mathrm{opt}}(\eta_i^*, \eta_{ij}) = \begin{cases} \dfrac{2}{(1-\eta_i^*)|\Omega_i|} \max\limits_{j\in\mathcal{V}(i)} \dfrac{|\Gamma_{ij}|}{\eta_{ij}}, & \text{if } \eta_i^* \geq \overline{\eta_i^*}, \\ \dfrac{|\partial\Omega_i|}{|\Omega_i|} \left[\min\limits_{j\in\mathcal{V}(i)} \left\{ \eta_i^* \left(1 - \dfrac{2|\Gamma_{ij}|}{|\partial \mathrm{T}_{ij}|} - \dfrac{\eta_{ij}|\partial\Omega_i|}{|\partial \mathrm{T}_{ij}|}\right) + \dfrac{\eta_{ij}|\partial\Omega_i|}{|\partial \mathrm{T}_{ij}|} \right\} \right]^{-1} \end{cases}$$

An optimal bound for the solution is given by

$$\mu_i^{\mathrm{opt}}\left(\eta_i^* = \frac{1}{3}, \eta_{ij} = \frac{|\Gamma_{ij}|}{|\partial\Omega_i|}\right) = 3\frac{|\partial\Omega_i|}{|\Omega_i|}.$$

Example




1/#V(i)
|T_ij|/|Ω_i|
|Γ_ij|/|∂Ω_i|
Optimal
250
200
150
100
50
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
μ_i^opt
η_i^*

Algorithm

1. Iteration 0: computation of η_{ij} and $\mu_i^{\mathtt{opt}}$
2. Iterations $n \geqslant 1$: computation of
   - local gradients $\Delta \mathbf{W}_{ij}^n$
   - intermediate gradient $\Delta \mathbf{W}_i^* = -\sum_{j \in \mathcal{V}(i)} \eta_{ij} \Delta \mathbf{W}_{ij}^n$
   - $\beta_i^n = \min\{1, \frac{1}{2}\vartheta_i^{(\rho)}, \frac{1}{2}\vartheta_i^{(p)}\}$
   - reconstructed states $\mathbf{W}_{ij}^n = \mathbf{W}_i^n + \beta_i^n \Delta \mathbf{W}_{ij}^n$ and

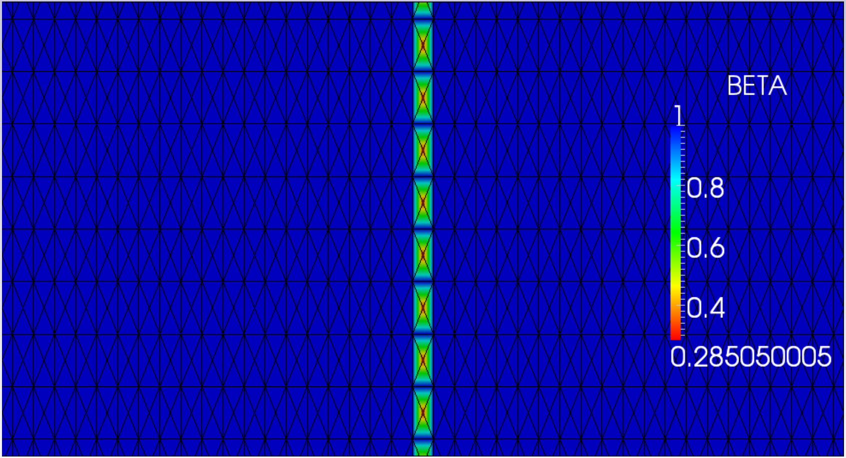
     $$\mathbf{W}_i^* = \mathbf{W}_i^n - 2\beta_i^n \sum_{j \in \mathcal{V}(i)} \eta_{ij} \Delta \mathbf{W}_{ij}^n$$

   - eigenvalues and time step

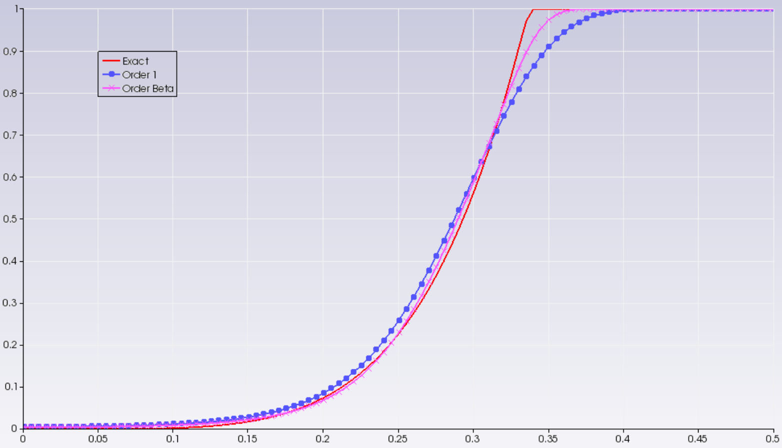
     $$\Delta t^n = \frac{\alpha_0}{\mu_i^{\mathtt{opt}} \bar{\lambda}_i^n}$$

   - updated state $\mathbf{W}_i^{n+1}$

1-2-3 test case

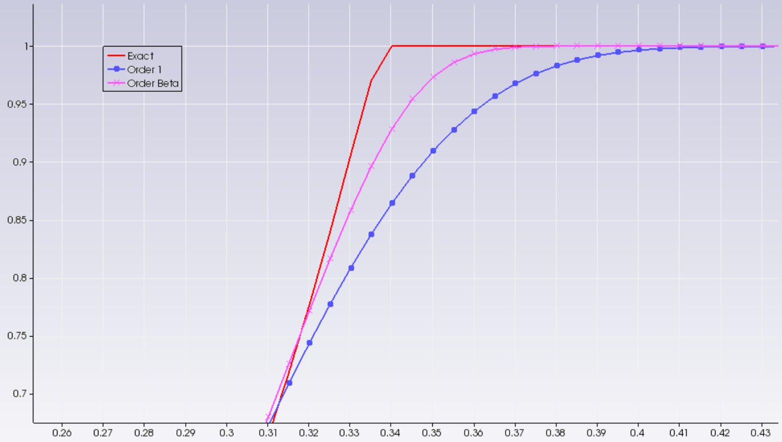


1-2-3 test case



1-2-3 test case

1-2-3 test case



Perspectives

Done

- ✓ Analysis of robustness of general MUSCL strategies
- ✓ Derivation of (sufficient) conditions to preserve positivity and entropy inequalities
- ✓ Explicit CFL conditions
- ✓ Direct extension to 2nd-order in time (RK2)
- ✓ Easy adaptation of industrial codes

C. Calgaro, E. Creusé, T. Goudon & Y. Penel, *Positivity-preserving schemes for Euler equations: sharp and practical CFL conditions* (J. Comput. Phys., 234, 2013).

To do

- ❧ Influence of the numerical flux on β_i^n
- ❧ Application to other systems of conservation laws

THANK YOU FOR YOUR ATTENTION