Exam

No document is authorized except the lecture notes. Cell phones, calculators and computers are forbidden. Answers can be given either in French or in English.
Seul le polycopié de cours est autorisé. Les téléphones portables, calculatrices et ordinateurs sont interdits. Les réponses peuvent être rédigées en français ou en anglais.

Exercise 1 *We aim at solving the standard ordinary differential equation*

$$\begin{cases} y'(t) = f\big(t, y(t)\big), & \text{(1a)} \\ y(0) = 2. & \text{(1b)} \end{cases}$$

by means of different numerical schemes. We assume that f is smooth enough so that ODE (1) *has a unique solution $\hat{y} \in \mathscr{C}^2(0,1)$ according to the* Cauchy-Lipschitz *theorem.*

Let us define the time discretization through

$$t^n = (n-1)\Delta t,\ 1 \le n \le N,\ \Delta t = \frac{1}{N-1}. \tag{2}$$

for some $N > 0$. The numerical approximation of the solution to (1a) *is denoted by (y_n).*

1. *Let us introduce the* 2-step Adams-Moulton *scheme defined by*

$$y_{n+2} - y_{n+1} = \Delta t \left(\frac{5}{12} f(t^{n+2}, y_{n+2}) + \frac{2}{3} f(t^{n+1}, y_{n+1}) - \frac{1}{12} f(t^n, y_n) \right). \tag{3}$$

   (a) *Prove the consistency of Scheme* (3).
   (b) *Show that this scheme is stable.*
   (c) *Can you deduce that the scheme is convergent? Explain briefly what that means.*
   (d) *Determine the order of the scheme.*
   (e) *What is t^1 equal to? t^2? t^N?*
   (f) *How many values do we need in order to initialize the sequence (y_n)?*
   (g) *Given the order of the scheme, what seems to be the most relevant way to compute these initializing values?*
   (h) *Is this scheme implicit or explicit?*

2. *We take in this question* $\boxed{f(t,z) = z}$.

   (a) *What is the exact solution to ODE* (1)*?*
   (b) *Apply Scheme* (3) *to this case.*
   (c) *Show that for any $N > 0$, it is possible to compute y_n as a function of n and Δt.*
   (d) *Apply the* 3rd-order Runge-Kutta *scheme to the present case. The corresponding sequence will be denoted by (z_n).*
   (e) *Compute z_n as a function of n and Δt.*
   (f) *Which scheme would you recommend:* Adams-Moulton *or* Runge-Kutta*? Justify your answer.*

Exercise 2

1. *Recall the statement of the* Cholesky *decomposition.*
2. *Write down a* MATLAB *function which provides the* Cholesky *decomposition of a given matrix A. The algorithm must be carefully designed.*
3. *Let A be the* 3×3 *matrix*

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

   (a) *Justify that matrix A has a* Cholesky *decomposition.*

   (b) *Compute this decomposition.*

4. *What is the* Cholesky *decomposition useful for?*
5. *Deduce the algorithm solving the linear system $Ax = b$ for a given vector b when A satisfies the hypotheses of the* Cholesky *statement.*