Exercises #3

Let $f : [0,1] \times \mathbb{R} \longmapsto \mathbb{R}$ be a smooth function of class $\mathscr{C}^1$. We aim at approximating the solution of the ODE

$$\hat{y}'(t) = f\big(t, \hat{y}(t)\big). \tag{1}$$

Let T be some positive number and $N \in \mathbb{Z}_+$, $N \neq 0$. Then we set $\Delta t = \frac{T}{N}$ and $t^n = n\Delta t$, $0 \leq n \leq N$.

Exercise 14 *We assume in this exercise that* $f(t,y) = -y$.

1. *Solve* (1) *in this case supplemented with the initial condition* $\hat{y}(0) = 1$.
2. *Apply the explicit Euler scheme to construct the sequence* (y_n).
3. *Yield the explicit expression of* y_n *with respect to* n *and* Δt*. Is the scheme relevant for any* Δt*?*
4. *Compare* y_N *and* $\hat{y}(T)$*. Conclude.*
5. *Follow the same directions about the Heun scheme.*
6. *Which scheme seems to be the most efficient?*

Exercise 15 *We take in this exercise* $f(t,y) = 1 - 2y$.

1. *Solve* (1) *in this case supplemented with the initial condition* $\hat{y}(0) = 1$.
2. *Apply the implicit Euler scheme to construct the sequence* (y_n).
3. *Yield the explicit expression of* y_n *with respect to* n *and* Δt.
4. *We now introduce the* θ*–scheme (for* $\theta \in [0,1]$*)*

$$\frac{y_{n+1} - y_n}{\Delta t} = \theta f(t^n, y_n) + (1-\theta) f(t^{n+1}, y_{n+1}).$$

   (a) *Apply this scheme to the present case and solve the inductive relation.*
   (b) *What happens for* $\theta = 0$*?*
   (c) *Let* f_θ *be the function*

$$f_\theta(\Delta t) = e^{-\Delta t} - \frac{1 - \theta\Delta t}{1 + (1-\theta)\Delta t}.$$

   Determine the value of θ *for which this function is the closest to* 0 *as* Δt *becomes small. What is the corresponding scheme?*

Exercise 16 *To provide an approximate solution to* (1)*, we propose the scheme*

$$\frac{3y_{n+2} - 4y_{n+1} + y_n}{2} = \Delta t\, f(t^{n+2}, y_{n+2}).$$

1. *How can this scheme be initialized?*
2. *Show that the scheme is convergent. Determine its order.*
3. *Is this scheme explicit?*
4. *In the case* $f(t,y) = -y$*, solve the linear inductive relation for* y_n.
5. *Propose a modification of the right hand side in the previous scheme to improve the order.*

Exercise 17 *The enhanced Euler scheme reads*

$$y_{n+1} = y_n + \Delta t f\big(t^{n+1}, y_n + \Delta t f(t^n, y_n)\big).$$

1. *Compute* $\hat{y}''(t)$ *for* $\hat{y}$ *solution of* (1).

2. *Show this scheme is convergent and determine its order.*

Exercise 18 *We assume that* (1) *is autonomous,* i.e. *that* f *does not depend on* t. *We consider the* lintrap *scheme*

$$\frac{y_{n+1} - y_n}{\Delta t} = f(y_n) + f'(y_n)\frac{y_{n+1} - y_n}{2} \tag{2}$$

1. *Show that* (2) *is a consistant scheme up to order* 2.

2. *We suppose in this question that* $f(y) = (y+1)^2$.

   (a) *Solve the ODE.*

   (b) *Apply Scheme* (2) *to* (1).

   (c) *Prove that the scheme is exact.*

Exercise 19 *Let us consider the system*

$$\begin{cases} x'(t) = y(t), \ x(0) = 1, \\ y'(t) = -x(t), \ y(0) = 0. \end{cases} \tag{3}$$

1. *Prove that the trajectories* $t \longmapsto \big(x(t), y(t)\big)$ *are included in the unit circle* $x^2 + y^2 = 1$.

2. *Write out the explicit Euler scheme, the implicit Euler scheme and the Crank-Nicholson scheme for the resolution of* (3).

3. *Do these schemes preserve the trajectories?*

Exercise 20 *Let us consider the ODE*

$$\begin{cases} \hat{y}'(t) + 3\hat{y}(t)^2 = 0, \\ \hat{y}(0) = 1. \end{cases} \tag{4}$$

1. *Solve* (4). *What is the limit of* $\hat{y}(t)$ *as* $t \to +\infty$*?*

2. *Apply the explicit Euler scheme to this ODE and study the limit as* $n \to +\infty$.

3. *We propose the following scheme*

$$\frac{y_{n+1} - y_n}{\Delta t} + 3\left(\frac{y_{n+1} + y_n}{2}\right)^2 = 0. \tag{5}$$

   (a) *Derive an expression of* y_{n+1} *with respect to* y_n *and deduce a condition upon* Δt *for the limit to be correct as* $n \to +\infty$.

   (b) *Show this scheme is consistant at order* 2.