Practical Work #3

The aim of this session is to apply numerical methods to different differential problems and to cope with classical numerical issues.

1 A nonlinear equation (exercise 11)

We consider the ODE

$$\begin{cases} t\hat{y}(t)\hat{y}'(t) = \hat{y}^2(t) - 1, \\ \hat{y}(1) = 2. \end{cases} \tag{1}$$

The exact solution $\hat{y}$ was calculated in exercise 11. Set $t^n = 1 + (n-1)\Delta t$, $1 \le n \le N$ and $\Delta t = \frac{2}{N-1}$.

1. What is the final time associated to the discretization?
2. Implement the explicit Euler scheme and the enhanced Euler scheme.
3. Show the curves $t \mapsto \big(t, y_{num}(t)\big)$ for $N = 100$, $N = 200$, $N = 500$ and $N = 1000$ as well as $t \mapsto \big(t, \hat{y}(t)\big)$.
4. For these four values of N, compute $\max\limits_{1 \le n \le N} |y_n - \hat{y}(t^n)|$. Plot these values for the two schemes and discuss their performance.

2 Lorenz model

The Lorenz model reads

$$\begin{cases} y_1' = 10(y_2 - y_1), \qquad y_1(0) = -8, \\ y_2' = 28y_1 - y_2 - y_1 y_3, \qquad y_2(0) = 8, \\ y_3' = y_1 y_2 - \frac{8}{3} y_3, \qquad y_3(0) = 27. \end{cases} \tag{2}$$

No exact solution is known for this system of ODEs. That is why we use numerical schemes to construct an approximation of the solution.

1. Implement the explicit Euler scheme as well as the RK4 scheme.
2. Plot the graphs $t \mapsto \big(t, y_1(t)\big)$, $t \mapsto \big(t, y_2(t)\big)$ and $t \mapsto \big(t, y_3(t)\big)$ for both schemes.
3. Plot the curve $t \mapsto \big(y_1(t), y_2(t)\big)$.
4. Compare the numerical solutions. What can you say about the behaviour of the solution?

3 Lotka–Volterra model (exercise 12)

In the Lotka–Volterra model

$$\begin{cases} x'(t) = x(t)\big(3 - y(t)\big), \qquad x(0) = 1, \\ y'(t) = y(t)\big(x(t) - 2\big), \qquad y(0) = 2. \end{cases} \tag{3}$$

x and y denote respectively the rate of preys and predators in a closed area. Even if the exact solution is not explicitly known, it is proven that x and y are periodic functions of time. Moreover, the Hamiltonian

$$H(x, y) = x - 2\ln x + y - 3\ln y$$

is preserved.

1. Implement the explicit Euler scheme and the Heun scheme.
2. Plot the graphs $t \mapsto \Big(t, H\big(x(t), y(t)\big)\Big)$ for both schemes as well as the theoretical value.
3. Draw $t \mapsto \big(x(t), y(t)\big)$. Is the result as expected?
4. We propose the following scheme

$$\begin{cases} \dfrac{x_{n+1}-x_n}{\Delta t} = x_n(3-y_n), \\ \dfrac{y_{n+1}-y_n}{\Delta t} = y_n(x_{n+1}-2). \end{cases}$$

   Is the conclusion of Question 3. still holds?