Exercises #2

Exercise 7 *Give the solutions to the following ODEs:*

$$\hat{y}'(t) = 2\hat{y}(t) + 1; \qquad \hat{y}'(t) = \frac{t}{t^2+1}\hat{y}(t); \qquad \hat{y}'(t) = -(\tan t)\hat{y}(t) - 1; \qquad \hat{y}'(t) = -\frac{\hat{y}(t)}{t} + \cos t.$$

Exercise 8 *Let* Ch *and* Sh *be the functions defined by*

$$\mathrm{Ch}(t) = \frac{e^t + e^{-t}}{2} \quad \textit{and} \quad \mathrm{Sh}(t) = \frac{e^t - e^{-t}}{2}.$$

1. *Compute the derivatives of* Ch *and* Sh.
2. *Give a particular solution y_0 of the differential system*
$$y'(t) - 2ty(t) = \mathrm{Sh}(t) - 2t\mathrm{Ch}(t). \tag{1}$$
3. *Characterize ODE* (1).
4. *Set $u = y - y_0$. Show that y is a solution to* (1) *together with $y(0) = 0$ if and only if u satisfies*
$$\begin{cases} u'(t) - 2tu(t) = 0, \\ u(0) = 1. \end{cases}$$
5. *Solve* (1) *together with $y(0) = 0$.*

Exercise 9

1. *Determine the solutions to the second-order ODE*
$$y''(t) - 2y'(t) + y(t) = 0.$$
2. *We then focus on the ODE*
$$\begin{cases} y''(t) - 2y'(t) + y(t) = \cos t, \\ y(0) = 0, \\ y'(0) = 1. \end{cases} \tag{2}$$
   (a) *Prove that there exists a unique solution to* (2).
   (b) *What are the eigenvalues of the corresponding matrix?*
   (c) *Determine C such that $y(t) = C(t)e^t$ satisfies* (2).
   (d) *Conclude.*

Exercise 10 *Let us solve the ODE*

$$\hat{y}'(t) = \frac{2}{e^{\hat{y}(t)} + e^{-\hat{y}(t)}}, \ \hat{y}(0) = 1.$$

1. *Prove there exists a unique solution over a certain interval J.*
2. *Set $F(x) = \ln\left(x + \sqrt{x^2+1}\right)$. Compute $F(\mathrm{Sh}\,t)$.*
3. *Give the expression of the solution according to the lecture notes.*

Exercise 11 *Solve the ODE*

$$ty'(t)y(t) = y(t)^2 - 1.$$

You may find the equation satisfied by $z = y^2$.

Exercise 12 *These equations model the evolution of an isolated predator-prey system (for instance rabbits and lynx):*

$$\begin{cases} x'(t) = x(t)\big(3 - y(t)\big), & x(0) = 1, \\ y'(t) = y(t)\big(x(t) - 2\big), & y(0) = 2. \end{cases} \tag{3}$$

1. *Determine which variable corresponds to the number of preys.*
2. *Determine the only constant solution to* (3).
3. *Rewrite Eqs.* (3) *as* $\mathbf{Y}'(t) = \mathbf{F}\big(\mathbf{Y}(t)\big)$, *where* $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$. *Deduce that there exists a unique solution* $\mathbf{Y}$.
4. *Prove that x and y cannot vanish.*
5. *We set* $H(x, y) = x - 2\ln x + y - 3\ln y$. *$H$ is called the* Hamiltonian *of the system. Show that for all* $t \geq 0$, $H\big(x(t), y(t)\big) = H\big(x(0), y(0)\big)$.

Exercise 13 *We get interested in a chemical experiment where a chemical species C is produced from A thanks to a catalyser B. The three reactions involved are*

$$A \longrightarrow B \qquad \textit{(slow)}$$
$$B + B \longrightarrow B + C \qquad \textit{(very fast)}$$
$$B + C \longrightarrow A + B \qquad \textit{(fast)}$$

The concentration of species $*$ is denoted by y_*. Their evolution is governed by the following ODEs:

$$\begin{cases} y_A'(t) = -k_1 y_A(t) + k_3 y_B(t) y_C(t), & \text{(4a)} \\ y_B'(t) = k_1 y_A(t) - k_3 y_B(t) y_C(t) - k_2 y_B^2(t), & \text{(4b)} \\ y_C'(t) = k_2 y_B^2(t), & \text{(4c)} \end{cases}$$

supplemented with the initial conditions

$$y_A(0) = 1, \; y_B(0) = 0, \; y_C(0) = 0. \tag{4d}$$

The reaction constants are $k_1 = 0.04$, $k_2 = 3 \cdot 10^7$ *and* $k_3 = 10^4$.

1. *Prove that there exists a unique solution to* (4) *on a certain interval J.*
2. *Prove that* $y_A(t) + y_B(t) + y_C(t) = 1$ *for all* $t \in J$.
3. *Determine the constant solutions to (4a-4b-4c).*
4. *Show that* $y_*(t) \geq 0$ *for all* $t \in J$ *and* $* \in \{A, B, C\}$.
5. *Deduce that* $J = \mathbb{R}$.