PW #4: FDM for the Black & Scholes model

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The Finite Difference Method (FDM) has been studied in the previous projects. We aim at applying this method to the well-known Black & Scholes equation with constant volatility σ and constant interest rate r for the modelling of a European vanilla put option:

$$\begin{cases} \dfrac{\partial \tilde{P}}{\partial t}(t,S) + \dfrac{\sigma^2 S^2}{2}\dfrac{\partial^2 \tilde{P}}{\partial S^2}(t,S) + rS\dfrac{\partial \tilde{P}}{\partial S}(t,S) - r\tilde{P}(t,S) = 0, \\ \tilde{P}(\mathcal{T},S) = \max(0, K-S), \end{cases}$$

or equivalently (by means of the change of variables $P(t,S) = \tilde{P}(\mathcal{T}-t,S)$):

$$\begin{cases} \dfrac{\partial P}{\partial t}(t,S) - \dfrac{\sigma^2 S^2}{2}\dfrac{\partial^2 P}{\partial S^2}(t,S) - rS\dfrac{\partial P}{\partial S}(t,S) + rP(t,S) = 0, & \text{(1a)} \\ P(0,S) = \max(0, K-S). & \text{(1b)} \end{cases}$$

We set for the present study:

$$K = 100,\ \mathcal{T} = 1,\ \sigma = 0.2 \text{ and } r = 0.04\,.$$

If P is a solution to (1), then the following function

$$\varphi(\theta,x) = P(\theta,e^x)e^{-\alpha\theta-\beta x} \qquad \text{with} \quad \beta = \frac{1}{2} - \frac{r}{\sigma^2},\ \alpha = -r - \frac{\sigma^2\beta^2}{2},$$

is a solution to:

$$\begin{cases} \dfrac{\partial \varphi}{\partial \theta} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 \varphi}{\partial x^2} = 0, & \text{(2a)} \\ \varphi(0,x) = e^{-\beta x}\cdot\max(0, K-e^x). & \text{(2b)} \end{cases}$$

We recall that the exact solution is given by:

$$\tilde{P}(t,S) = Ke^{-r(\mathcal{T}-t)}\Phi(-d_2) - S\Phi(-d_1),$$

with $\Phi(d) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{d} \exp\left(-\frac{z^2}{2}\right) \mathrm{d}z$ and:

$$d_1 = \frac{\ln(S/K) + (r+\sigma^2/2)(\mathcal{T}-t)}{\sigma\sqrt{\mathcal{T}-t}},\ d_2 = d_1 - \sigma\sqrt{\mathcal{T}-t}.$$

We thus aim at simulating equivalent formulations (1) and (2) and then at comparing corresponding solutions.

Exercise 1 (*Heat equation on a uniform grid*)

Let $\underline{x}$ and $\overline{x}$ be two real numbers such that $\underline{x} \ll \ln K \ll \overline{x}$. We consider a space discretization given by $x_j = \underline{x} + (j-1)\Delta x$, $\Delta x = \dfrac{\overline{x}-\underline{x}}{N_x - 1}$ for some $N_x > 1$ and a time discretization $t_n = (n-1)\Delta t$, $\Delta t = \dfrac{\mathcal{T}}{N_t - 1}$ for a suitable $N_t > 1$.

Implement and compare performance of the explicit Euler scheme, the implicit Euler scheme and the Crank-Nicholson scheme for the resolution of (2). Comparisons will be made on the primitive function $\tilde{P}$. Note that the stability condition reads $\Delta t \leq \dfrac{\Delta x^2}{\sigma^2}$.

Exercise 2 (*Heat equation on a nonuniform grid*)

Let $\underline{S}$ and $\overline{S}$ be two real numbers such that $\underline{S} \ll K \ll \overline{S}$. We consider an asset discretization given by $S_j = \underline{S} + (j-1)\Delta S$, $\Delta S = \dfrac{\overline{S}-\underline{S}}{N_S - 1}$ for some $N_S > 1$ and a time discretization $t_n = (n-1)\Delta t$, $\Delta t = \dfrac{\mathcal{T}}{N_t - 1}$ for some $N_t > 1$. The corresponding space discretization is imposed by the change of variable:

$$x_j = \ln S_j.$$

We set $\Delta x_j = x_{j+1} - x_j$.

1. *Derive a formula approaching the second order spatial derivative on nonuniform grids.*
2. *Then implement the resolution of (2) by means of the Crank-Nicholson scheme.*

Exercise 3 (*Resolution of the Black & Scholes model in primitive variables*)

1. *Given a uniform discretization of the time-asset space $[0, \mathcal{T}] \times [\underline{S}, \overline{S}]$, propose a numerical scheme based on the explicit Euler scheme for the time derivative to solve (1).*
2. *Find out by means of numerical simulations a suitable value for Δt.*
3. *Implement the Crank-Nicholson scheme applied to (1).*
4. *Comment numerical results.*