Application of an AMR technique to an Abstract Bubble Vibration Model

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INTRODUCTION

GOAL

Modeling of the evolution of bubbles in a nuclear reactor at the scale of bubbles (DNS)

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Navier-Stokes equations for a diphasic compressible immiscible flow

Mach Number supposed to be small (still compressible flow but without acoustic wave effects)

Diphasic Low Mach Number system (DLMN)

Simplification but keeping a similar mathematical structure

Abstract Bubble Vibration model (ABV)

Numerical representation of bubble interfaces

Numerical scheme for a transport equation

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Numerical representation of bubble interfaces

→ Numerical scheme for a transport equation

AMR

OUTLINE

Partial Outline

Compressible Diphasic Navier-Stokes System

Starting point

Diphasic Compressible Navier-Stokes

$$\begin{cases} \partial_t(\rho Y_1) + \nabla\cdot(\rho Y_1 \mathbf{u}) = 0\,, \\ \partial_t \rho + \nabla\cdot(\rho\mathbf{u}) = 0\,, \\ \partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) = -\nabla P + \nabla\cdot\sigma + \rho\mathbf{g}\,, \\ \partial_t(\rho E) + \nabla\cdot(\rho\mathbf{u}E) = -\nabla\cdot(P\mathbf{u}) + \nabla\cdot(\kappa\nabla T) + \nabla\cdot(\sigma\mathbf{u}) + \rho\mathbf{g}\cdot\mathbf{u}\,, \end{cases}$$

together with transmission and boundary conditions.

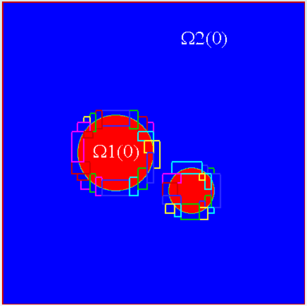
Nomenclature

- Y_1 : mass fraction of Fluid 1
- T : temperature
- P : pressure
- $\mathbf{u}$: global velocity
- E : total energy
- ρ : density
- $\mathbf{g}$: gravity field
- σ : Cauchy stress tensor
- κ : thermal conductivity
- $\theta = (Y_1, T, P)$

Modeling bubbles

Initial Condition

$$Y_1(t=0,x) = Y^0(x) = \begin{cases} 1, & \text{si } x \in \Omega_1(0), \\ 0, & \text{si } x \in \Omega_2(0). \end{cases}$$



As Y_1 is solution of the equation $\partial_t(\rho Y_1) + \nabla\cdot(\rho Y_1 \mathbf{u}) = 0 \Longleftrightarrow \partial_t Y_1 + \mathbf{u}\cdot\nabla Y_1 = 0$, the interface of the bubble coincides with its discontinuity. The resolution of this equation with that initial condition amounts to determining for a domain $\Omega_1(t)$.

Hypotheses and Tools

Physics	Mathematics
Bounded domain (reactor)	$\Omega = [-1,1]^d$, $d \in \{2,3\}$
No void	$\rho > 0$
Linear elasticity	Linearized Cauchy tensor
Common physical properties for both fluids	Single nondimensioned system
Low Mach Number	Asymptotic expansion w.r.t. $\mathscr{M}_* \ll 1$

Based on earlier Majda's & Embid's works on combustion ('84).

DLMN System

Diphasic Low Mach Number System: At order 0 in the asymptotic expansion, the system reads:

$$\begin{cases} \partial_t Y_1 + \mathbf{u}\cdot\nabla Y_1 = 0\,, \\ \nabla\cdot\mathbf{u} = G_\theta\,, \\ \rho(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\nabla\Pi + 2\nabla\cdot[\mu D(\mathbf{u})] + \rho\mathbf{g}\,, \\ \rho c_p(\partial_t T + \mathbf{u}\cdot\nabla T) = \alpha T P'(t) + \nabla\cdot(\kappa\nabla T)\,, \\ P'(t) = H_\theta(t), \end{cases}$$

where P is the thermodynamic pressure, Π the dynamic pressure and:

$$G_\theta(t,x) := -\frac{D_t\rho}{\rho} = -\frac{1}{\Gamma}\frac{P'(t)}{P(t)} + \frac{\beta\nabla\cdot(\kappa\nabla T)}{P(t)},$$

$$H_\theta(t) := \frac{\displaystyle\int_\Omega \beta(\theta)\nabla\cdot\left[\kappa(\theta)\nabla T\right]\mathrm{d}\mathbf{x}}{\displaystyle\int_\Omega \frac{1}{\Gamma(\theta)}\,\mathrm{d}\mathbf{x}}.$$

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$$\begin{cases} \partial_t Y_1 + \mathbf{u}\cdot\nabla Y_1 = 0\,, \\ \nabla\cdot\mathbf{u} = G_\theta\,, \qquad \overset{\neq 0}{\Longrightarrow} \qquad \text{compressibility, elliptic contribution} \\ \rho(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\nabla\Pi + 2\nabla\cdot[\mu D(\mathbf{u})] + \rho\mathbf{g}\,, \\ \rho c_p(\partial_t T + \mathbf{u}\cdot\nabla T) = \alpha T P'(t) + \nabla\cdot(\kappa\nabla T)\,, \\ P'(t) = H_\theta(t), \end{cases}$$

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Derived Systems for prelimary studies

Hodge Decomposition: $\mathbf{u} = \nabla\phi + \mathbf{w}$ with $\nabla\cdot\mathbf{w} = 0$ and boundary conditions.

Potential DLMN System $\mathbf{w} \to 0$

$$\begin{cases} \partial_t Y_1 + \nabla\phi\cdot\nabla Y_1 = 0\,, \\ \Delta\phi = G_\theta\,, \\ \rho c_p(\partial_t T + \nabla\phi\cdot\nabla T) = \alpha T P'(t) + \nabla\cdot(\kappa\nabla T)\,, \\ P'(t) = H_\theta(t). \end{cases}$$

Abstract Bubble Vibration Model $G_\theta \to G_Y$

$$\begin{cases} \partial_t Y_1 + \nabla\phi\cdot\nabla Y_1 = 0\,, \\ \Delta\phi(t,\mathbf{x}) = \psi(t)\left[Y_1(t,\mathbf{x}) - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega Y_1(t,\mathbf{y})\,\mathrm{d}\mathbf{y}\right]. \end{cases}$$

Partial Outline

Existence and Uniqueness for DLMN

Diphasic Low Mach Number

$$\begin{cases} \partial_t Y_1 + \mathbf{u}\cdot\nabla Y_1 = 0\,, \\ \nabla\cdot\mathbf{u} = G_\theta\,, \\ \rho(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\nabla\Pi + 2\nabla\cdot[\mu D(\mathbf{u})] + \rho\mathbf{g}\,, \\ \rho c_p(\partial_t T + \mathbf{u}\cdot\nabla T) = \alpha T P'(t) + \nabla\cdot(\kappa\nabla T)\,, \\ P'(t) = H_\theta(t)\,. \end{cases}$$

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Theorem (Penel, '09)

Assuming $s > \left\lfloor \frac{d}{2} \right\rfloor + 4$, $\theta_0 \in \mathscr{H}^s$ such that, for all $x \in \mathbb{T}^d$, $\theta_0(x) \in G_0$ with $\overline{G_0} \subset \Theta$, and $\mathbf{u}_0 \in \mathscr{H}^{s-1}$, there exists $\mathscr{T} = \mathscr{T}(\|\theta_0\|_s, \|\mathbf{u}_0\|_s) > 0$ such that there exists a unique classical solution $(\theta, \mathbf{u}, \nabla\pi)$ to the DLMN system:

- $T, P \in \mathscr{X}_{s,\mathscr{T}}(\mathbb{T}^d)$, $Y_1 \in \mathscr{X}_{s-1,\mathscr{T}}(\mathbb{T}^d)$, $\dfrac{\partial\theta}{\partial t} \in \mathscr{X}_{s-2,\mathscr{T}}(\mathbb{T}^d)$;
- $\mathbf{u} \in \mathscr{X}_{s-1,\mathscr{T}}(\mathbb{T}^d)$, $\dfrac{\partial\mathbf{u}}{\partial t} \in \mathscr{X}_{s-3,\mathscr{T}}(\mathbb{T}^d)$;
- $\nabla\pi \in \mathscr{X}_{s-3,\mathscr{T}}(\mathbb{T}^d)$.

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$$\mathscr{X}_{s,\mathscr{T}}(\mathbb{T}^d) = \mathscr{C}^0\left([0,\mathscr{T}], L^2(\mathbb{T}^d)\right) \cap L^\infty\left([0,\mathscr{T}], \mathscr{H}^s(\mathbb{T}^d)\right) \cap L^2\left([0,\mathscr{T}], \mathscr{H}^{s+1}(\mathbb{T}^d)\right)$$

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Existence and Uniqueness for ABV

Abstract Bubble Vibration Model

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Theorem (Lafitte & Dellacherie, '05)

Assuming $s > \left\lfloor \frac{d}{2} \right\rfloor + 2$, $Y^0 \in \mathscr{H}^s$. There exists $\mathscr{T} = \mathscr{T}(\|Y^0\|_s) > 0$ such that there exists a unique classical solution $(Y_1, \nabla\phi)$ to the ABV system, with $Y_1 \in \mathscr{C}^0\left([0,\mathscr{T}], L^2(\mathbb{T}^d)\right) \cap L^\infty\left([0,\mathscr{T}], \mathscr{H}^s(\mathbb{T}^d)\right)$.

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Lemma

Assuming there exists a weak solution of the form $Y_1(t,\mathbf{x}) = \mathbf{1}_{\Omega_1(t)}(\mathbf{x})$ where $\Omega_1(t)$ is a smooth open set in Ω, and $\psi \in \mathscr{C}^0(0,+\infty)$, then the volume of the bubble is given by:

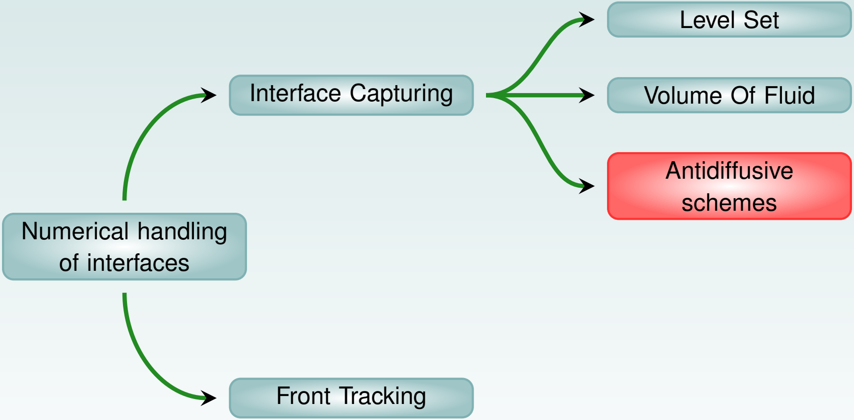
$$|\Omega_1(t)| = |\Omega| \frac{|\Omega_1(0)| \exp \int_0^t \psi(\tau)\mathrm{d}\tau}{|\Omega_2(0)| + |\Omega_1(0)| \exp \int_0^t \psi(\tau)\mathrm{d}\tau}.$$

Partial Outline

1. Derivation of the model
2. Theoretical results
3. Interfaces: Numerical resolution of a transport equation
   - Representation of interfaces
   - AMR
   - Numerical results
4. Numerical Results for the ABV model
5. Conclusion

INTERFACES

DL



INTERFACES

DL

Resolution of the transport equation $\partial_t Y_1 + \mathbf{u} \cdot \nabla Y_1 = 0$ by means of the Després-Lagoutière Scheme

B. Després, F. Lagoutière, *Contact Discontinuity Capturing Schemes for Linear Advection and Compressible Gas Dynamics.* J. of Sc. Comp., '01.

Idea: combining stability properties of the upwind scheme and non-diffusive properties of the downwind scheme.

Properties:

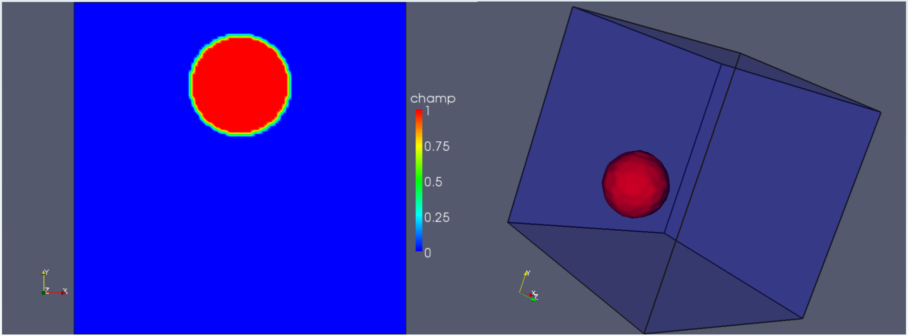
- L^∞-stable under CFL
- TVD
- uniform control of the number of diffusion cells

Antidiffusive schemes

FUNDAMENTAL TEST (KOTHE & RIDER)

Data

Resolution of $\partial_t Y_1 + \mathbf{u} \cdot \nabla Y_1 = 0$ with a periodic rotational prescribed velocity in 2D and 3D cases. The velocity is such that one should recover the initial condition after one period.



Improving accuracy requires additional techniques like AMR.

Some references

- **General concept of AMR** (Grouping-Clustering Algorithm)
  - M. Berger and J. Oliger, *Adaptive methods for hyperbolic partial differential equations.* JCP, '84.
  - M. Berger and P. Colella, *Local adaptive mesh refinement for shock hydrodynamics.* JCP, '89.
  - M. Berger and I. Rigoutsos, *An Algorithm for Point Clustrering and Grid Generation.* IEEE, '91.
- **AMR techniques for the transport equation**
  - J. Quirk, *An Adaptive Grid Algorithm for Computational Shock Hydrodynamics.* Ph.D. Thesis, Cranfield Univ., '91.
  - J.-C. Jouhaud, *Méthode d'Adaptation de Maillages Structurés par Enrichissement.* Ph.D. Thesis, Bordeaux Univ., '97.
  - J. Ryan and M. Borrel, *Adaptive Mesh Refinement: a coupling Framework for Direct Numerical Simulation of reacting Gas Flow.* ICFD, '04.
- **Local Defect Correction Methods for the Laplace equation**
  - W. Hackbush, *Local defect correction and domain decomposition techniques.* Defect Corr. Meth., '84.
  - M. Anthonissen, *Local Defect Correction Techniques: Analysis and Application to Combustion.* Ph.D. Thesis, Eindhoven Univ., '01.

GENERATING A HIERARCHICAL STRUCTURE OF GRIDS

Steps

- tagging grid regions that need a higher resolution
- clustering tagged cells into subgrids (patches)
- refining patches

Balance between general constraints

- as few patches as possible to reduce computation
- patches as small as possible to avoid unrelevant refined regions

***Grouping-Clustering* Algorithm**

- coarse level grid: G_0
- moving patches $G_{1,n}$ adapting to the solution of eq. $\partial_t Y_1 + \mathbf{u} \cdot \nabla Y_1 = 0$

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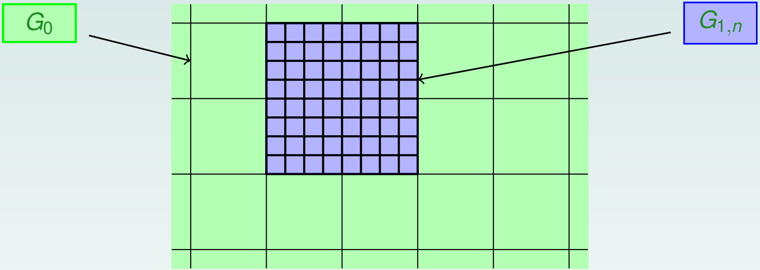

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- coarse level grid: G_0
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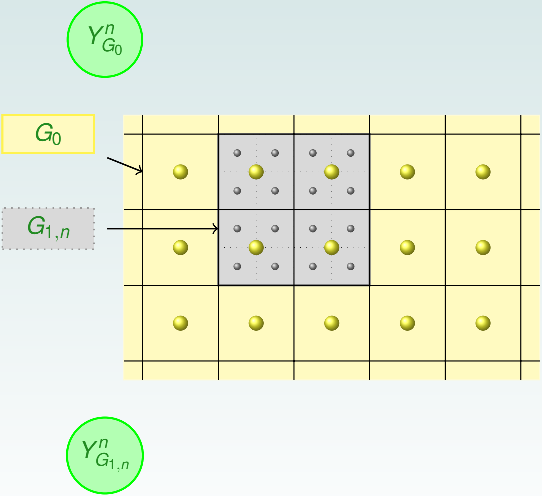
SKETCH OF RESOLUTION

Sketch of resolution



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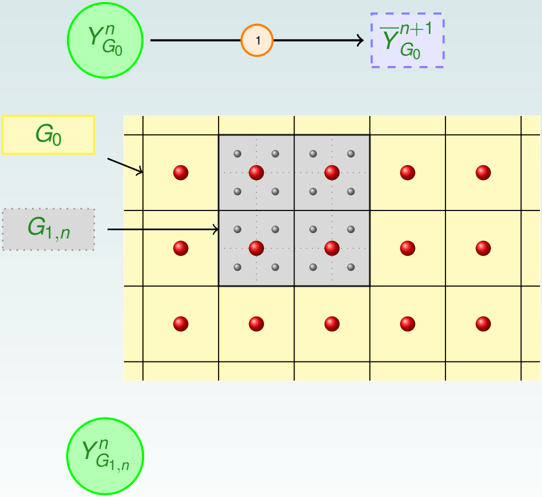

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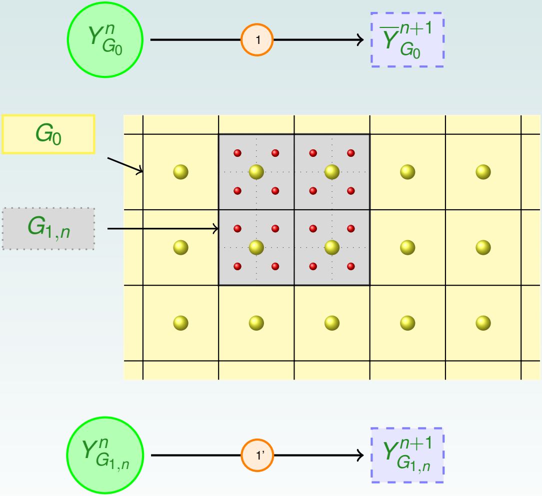

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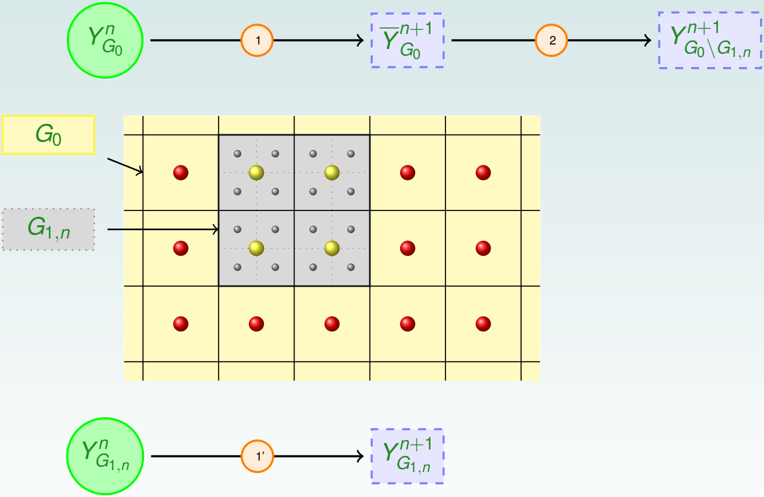

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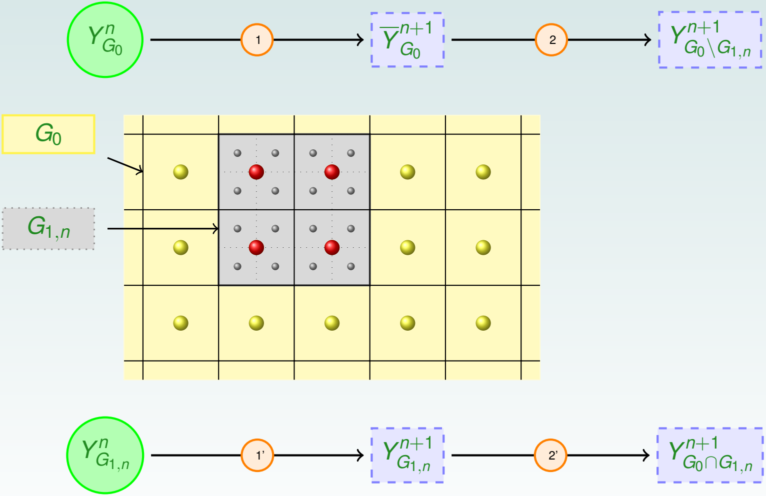

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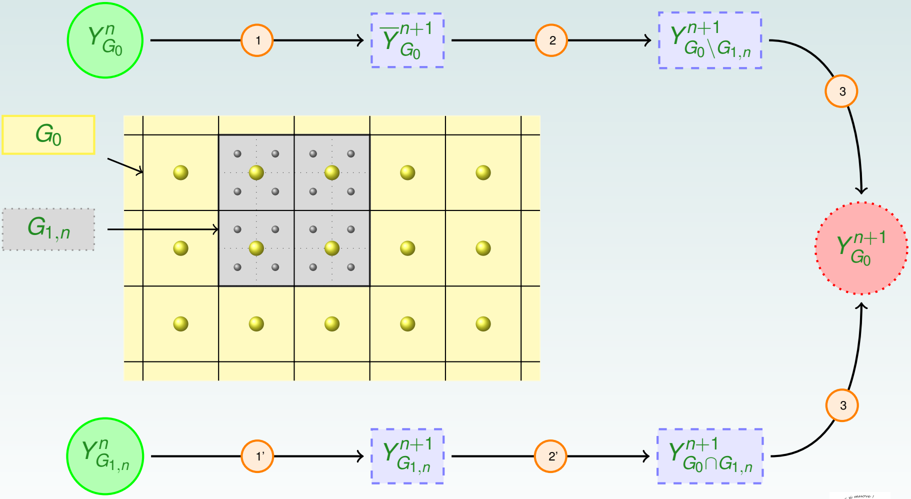

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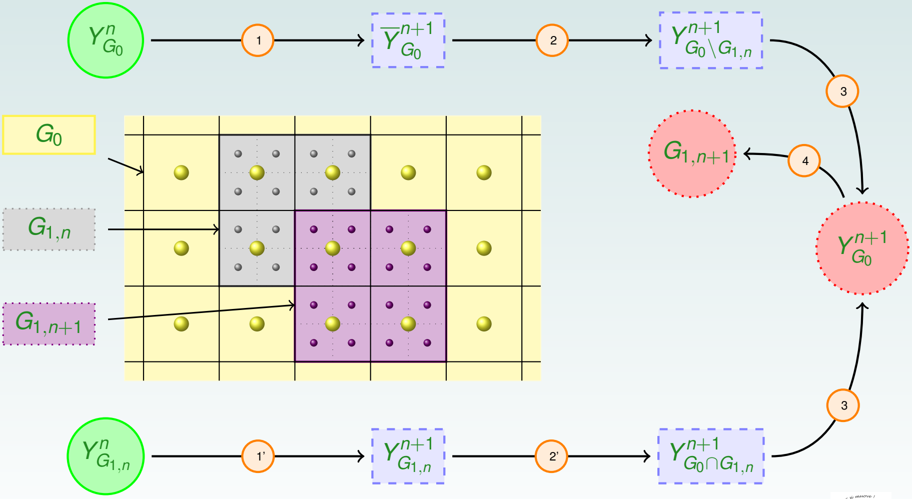

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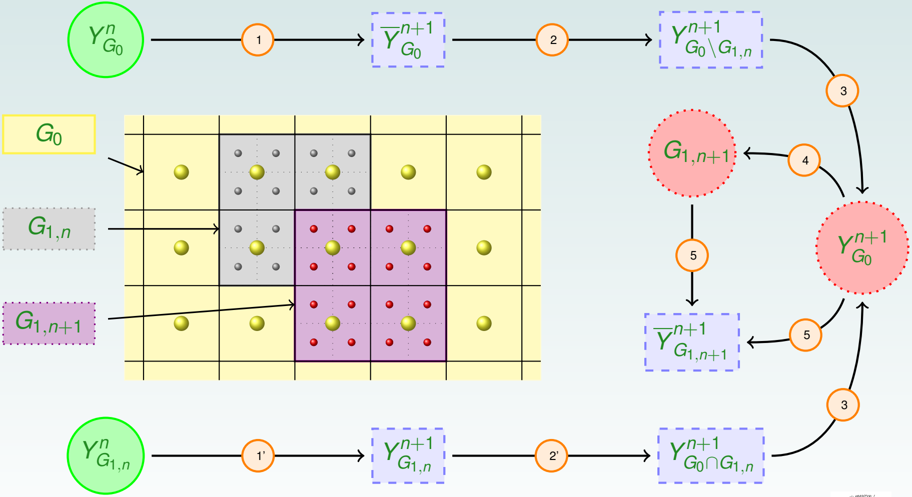

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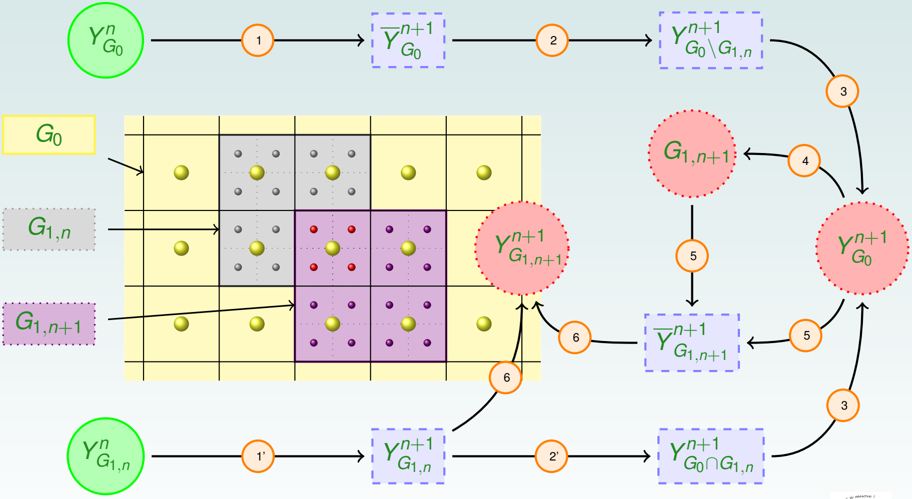
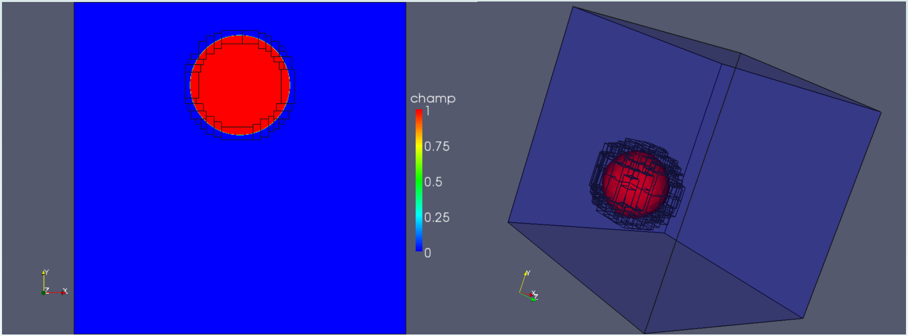


FUNDAMENTAL TEST (WITH AMR)

Data



Refinement rate equal to 10

Partial Outline

CODE STRUCTURE

LDC

$$\begin{cases} \partial_t Y_1 + \nabla\phi \cdot \nabla Y_1 = 0, \\ \Delta\phi = \mathscr{P}(Y_1). \end{cases}$$

Code structure

LDC

$$Y_{G_0}^n,\ G_{1,n},\ Y_{G_{1,n}}^n,\ \phi^n$$

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$Y_{G_0}^n$, $G_{1,n}$, $Y_{G_{1,n}}^n$, ϕ^n

DL-Scheme with AMR

for one time iteration of the resolution of the transport equation with discrete velocity $\nabla\phi^n$

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CODE STRUCTURE

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DL-Scheme with AMR

for one time iteration of the resolution of the transport equation with discrete velocity $\nabla\phi^n$

$$\begin{cases} \partial_t Y_1 + \nabla\phi \cdot \nabla Y_1 = 0, \\ \Delta\phi = \mathscr{P}(Y_1). \end{cases}$$

$Y^{n+1}_{G_0}$, $G_{1,n+1}$, $Y^{n+1}_{G_{1,n+1}}$

Code structure

LDC

$Y^n_{G_0}$, $G_{1,n}$, $Y^n_{G_{1,n}}$, ϕ^n

DL-Scheme with AMR

for one time iteration of the resolution of the transport equation with discrete velocity $\nabla\phi^n$

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LDC method

(using fine grid built in the hyperbolic step) for the resolution of the Laplace equation with rhs $\mathscr{P}(Y^{n+1})$

$Y^{n+1}_{G_0}$, $G_{1,n+1}$, $Y^{n+1}_{G_{1,n+1}}$

Code structure

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$$Y_{G_0}^{n+1},\ G_{1,n+1},\ Y_{G_{1,n+1}}^{n+1},\ G_{1,n+1}$$

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NUMERICAL TEST

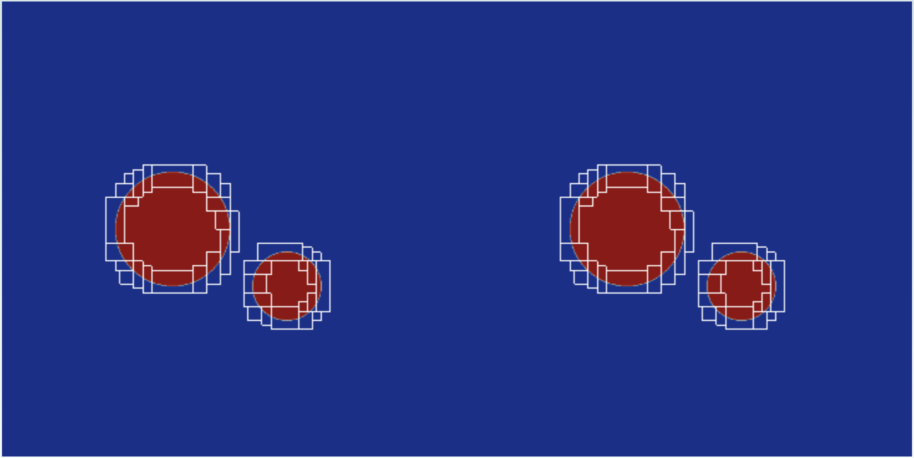
ABV model:

$$\begin{cases} \partial_t Y_1 + \nabla\phi \cdot \nabla Y_1 = 0, \\ \quad Y_1(0,\mathbf{x}) = Y^0(\mathbf{x}), \\ \Delta\phi = Y_1 - \dfrac{1}{4}\displaystyle\iint_{[-1,1]^2} Y_1(t,\mathbf{y})\,\mathrm{d}\mathbf{y}, \\ \quad \nabla\phi \cdot \mathbf{n}_{|\partial\Omega} = 0. \end{cases}$$

$\psi \equiv 1$ implies constant growth.

The initial condition Y^0 is defined by two circles with different radii.

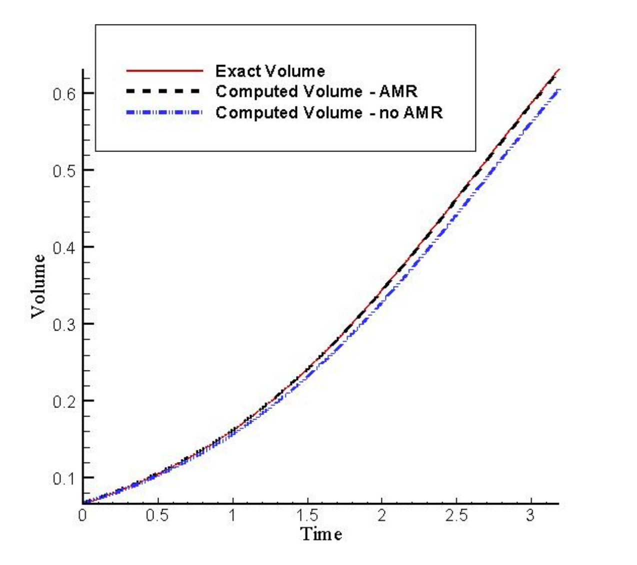
NUMERICAL TEST



100×100 grid with a refinement rate equal to 6

VOLUMES

▸ Lemma



Partial Outline

Conclusion and Prospects

Done

- Implementation of an AMR technique for a transport equation
- Coupling between hyperbolic and elliptic
- Coupling between three algorithms (DL scheme, AMR, LDC)

To do

- Application to the DLMN System
- Enrichment of the model with physical aspects
- Theoretical studies for non-smooth initial conditions

Thank you for your attention

Partial Outline

Definition of the scheme

Set:

$$\begin{cases} b_i^n(u) := \dfrac{Y_i^n - M_{i-1/2}}{u\Delta t/\Delta x} + M_{i-1/2}, \\ B_i^n(u) := \dfrac{Y_i^n - m_{i-1/2}}{u\Delta t/\Delta x} + m_{i-1/2}, \end{cases}$$

with $m_{i-1/2} = \min(Y_{i-1}^n, Y_i^n)$ and $M_{i-1/2} = \max(Y_{i-1}^n, Y_i^n)$.

Then the fluxes are defined by:

$$Y_{i+1/2}^n = \begin{cases} b_i^n(u) & \text{if } Y_{i+1}^n \leq b_i^n(u), \\ Y_{i+1}^n & \text{if } b_i^n(u) < Y_{i+1}^n < B_i^n(u), \\ B_i^n(u) & \text{if } B_i^n(u) \leq Y_{i+1}^n. \end{cases}$$

Comparison between DL and upwind schemes

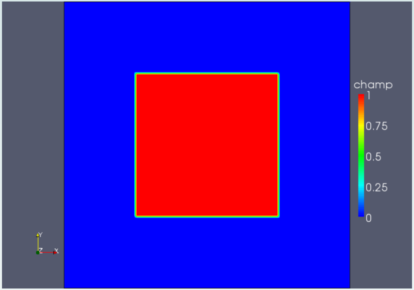


Schéma Després - Lagoutière

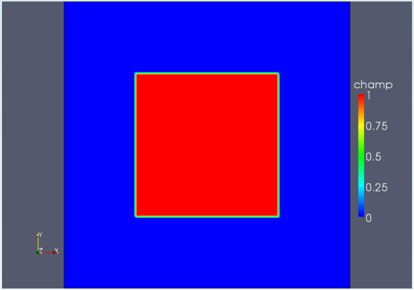


Schéma Upwind

Data

$$\mathbf{u}(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$\Omega = [-1,1]^2$$

$$\Delta x = \Delta y = \frac{1}{200}$$

Partial Outline

1D LDC method for $-\Delta\Phi = f$, with Neumann BC

Step 0

- Coarse grid resolution (modified conjugate gradient)

$$\begin{cases} \dfrac{\Phi_{H,0}(x_{i+1}) - 2\Phi_{H,0}(x_i) + \Phi_{H,0}(x_{i-1})}{H^2} = f_H(x_i), \quad i = \{1, \ldots, N-1\}, \\ \Phi'_{H,0}(x_0) = 0, \quad \Phi'_{H,0}(x_N) = 0; \end{cases}$$

- Fine grid resolution with updated Dirichlet BC (CG)

$$\begin{cases} \dfrac{\Phi_{h,0}(x_{i+1}) - 2\Phi_{h,0}(x_i) + \Phi_{h,0}(x_{i-1})}{h^2} = f_h(x_i), \qquad i = \{L_1, \ldots, L_2\}, \\ \Phi_{h,0}(x_{L_1}) = \Phi_{H,0}(x_{L_1}), \quad \Phi_{h,0}(x_{L_2}) = \Phi_{H,0}(x_{L_2}). \end{cases}$$

1D LDC method for $-\Delta\Phi = f$, with Neumann BC (2)

Step $k \geq 1$

- Coarse grid resolution with updated rhs

$$\begin{cases} \dfrac{\Phi_{H,k}(x_{i+1}) - 2\Phi_{H,k}(x_i) + \Phi_{H,k}(x_{i-1})}{H^2} = f_H(x_i) + R_{H,k-1}(x_i), \\ \Phi'_{H,k}(x_0) = 0, \quad \Phi'_{H,k}(x_N) = 0, \end{cases}$$

with

$$R_{H,s}(x_i) = \begin{cases} \dfrac{\Phi_{h,s}(x_{i+1}) - 2\Phi_{h,s}(x_i) + \Phi_{h,s}(x_{i-1})}{H^2} - f_H(x_i), \quad i = \{L_1, \ldots, L_2\}, \\ 0, \quad i = \{1, \ldots, L_1 - 1\} \cup \{L_2 + 1, \ldots, N\}; \end{cases}$$

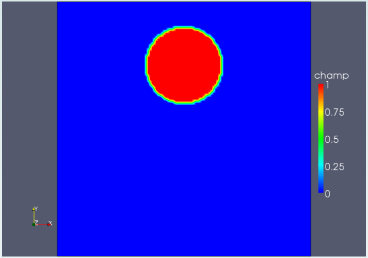
- Fine grid resolution

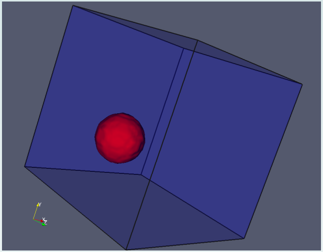
$$\begin{cases} \dfrac{\Phi_{h,k}(x_{i+1}) - 2\Phi_{h,k}(x_i) + \Phi_{h,k}(x_{i-1})}{h^2} = f_h(x_i), \\ \Phi_{h,k}(x_{L_1}) = \Phi_{H,k}(x_{L_1}), \quad \Phi_{h,k}(x_{L_2}) = \Phi_{H,k}(x_{L_2}). \end{cases}$$

Partial Outline

Institut Galilée

Data from Kothe - Rider tests





$$\mathbf{u}(t,x,y) = 2\cos\left(\frac{2\pi t}{T}\right)\sin(\pi x)\sin(\pi y)\times \begin{pmatrix} -\sin(\pi x)\cos(\pi y) \\ \sin(\pi y)\cos(\pi x) \end{pmatrix}$$

$$\Omega = [0,1]x[0,1]$$
$$\Delta x = \Delta y = \frac{1}{128}$$
$$T = 14$$

$$\mathbf{u}(t,x,y,z) = \cos\left(\frac{2\pi t}{T}\right)\times \begin{pmatrix} 2\sin^2(\pi x)\sin(2\pi y)\sin(2\pi z) \\ -\sin^2(\pi y)\sin(2\pi x)\sin(2\pi z) \\ -\sin(2\pi x)\sin(2\pi y)\sin^2(\pi z) \end{pmatrix}$$

$$\Omega = [0,1]x[0,1]x[0,1]$$
$$\Delta x = \Delta y = \Delta z = \frac{1}{64}, T = 6$$