Project #2:

semi-Lagrangian methods for solving the *Vlasov-Poisson* equation

In this project, we get interested in the *Vlasov-Poisson* equation:

$$\begin{cases} \partial_t f + \nabla_x \cdot (f\boldsymbol{v}) + \nabla_v \cdot (f\mathbf{E}) = 0, & \text{(1a)} \\ \mathbf{E}(t,\boldsymbol{x}) = -\nabla\phi(t,\boldsymbol{x}), & \text{(1b)} \\ -\Delta\phi(t,\boldsymbol{x}) = \displaystyle\int_{\mathbb{R}^d} f(t,\boldsymbol{x},\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} - 1, & \text{(1c)} \end{cases}$$

for which we propose several numerical methods. We assume **spatial periodicity** for the sake of simplicity. Only will the 1D case be studied in the sequel. For more details on theses methods, you can refer to the paper *Conservative numerical schemes for the Vlasov equation* by Filbet, Sonnendrücker and Bertrand (2001).

The first paragraph is related to an interpolation method called B-splines which is commonly used in semi-Lagrangian methods and known to be less diffusive than classical Lagrange interpolations. The very core of the project is the second part where we present three methods for solving (1).

1 B-spline interpolation

Unlike Lagrange interpolation, B-spline basis functions are not polynomials but piecewise polynomial functions. More precisely, for $N \in \mathbb{Z}_+$ and a uniform mesh $x_i = i\Delta x$, we expand the interpolated function as:

$$f(x) \approx \tilde{f}(x) := \sum_{i\in\mathbb{Z}} s_i^N(f)\mathcal{S}^N(x - x_i),$$

where $\mathcal{S}^N$ is a $\mathscr{C}^{N-1}$-function such that $(\mathcal{S}^N)_{|(x_j,x_{j+1})} \in \mathbb{R}_N[X]$. Functions $(\mathcal{S}^N)_N$ are computed from the convolution:

$$\mathcal{S}^0(x) = \frac{1}{\Delta x}\mathbf{1}_{(-\Delta x/2,\Delta x/2)}(x), \text{ then } \mathcal{S}^{N+1} = \mathcal{S}^N * \mathcal{S}^0.$$

Expansion coefficients are computed by taking into account interpolation requirements $f(x_i) = \tilde{f}(x_i)$ and boundary conditions.

1. Work out the expressions of $\mathcal{S}^1$, $\mathcal{S}^2$ and $\mathcal{S}^3$.
2. We get interested in the cubic case for periodic boundary conditions (which boils down to only considering n_x nodes). Show that $\left(s_i^3(f)\right)_i$ is prescribed by a linear system. Prove the matrix is invertible.
3. For $f(x) = \dfrac{1}{1+x^2}$, compare B-spline and Lagrange interpolations for different mesh sizes.

2 Semi-Lagrangian methods

These three methods are based on the methods of characteristics. Indeed, Eq. (1a) also reads:

$$\partial_t f + \boldsymbol{v} \cdot \nabla_x f + \mathbf{E} \cdot \nabla_v f = 0. \tag{2}$$

Hence, introducing the characteristic fields $\mathcal{X}$ and $\mathcal{V}$ defined as the solutions to:

$$\begin{cases} \dfrac{\mathrm{d}\mathcal{X}}{\mathrm{d}t} = \mathcal{V}(t; s, \boldsymbol{x}, \boldsymbol{v}), & \text{(3a)} \\ \dfrac{\mathrm{d}\mathcal{V}}{\mathrm{d}t} = \mathbf{E}\big(t, \mathcal{X}(t; s, \boldsymbol{x}, \boldsymbol{v})\big), & \text{(3b)} \\ \mathcal{X}(s; s, \boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{x}, \quad \mathcal{V}(s; s, \boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{v}, & \text{(3c)} \end{cases}$$

leads to:

$$f(s, \boldsymbol{x}, \boldsymbol{v}) = f\big(t, \mathcal{X}(t; s, \boldsymbol{x}, \boldsymbol{v}), \mathcal{V}(t; s, \boldsymbol{x}, \boldsymbol{v})\big).$$

A natural numerical scheme inferred from this formula is:

$$f(t^{n+1}, \boldsymbol{x}_i, \boldsymbol{v}_j) = f\big(t^n, \mathcal{X}(t^n; t^{n+1}, \boldsymbol{x}_i, \boldsymbol{v}_j), \mathcal{V}(t^n; t^{n+1}, \boldsymbol{x}_i, \boldsymbol{v}_j)\big). \tag{3d}$$

The problem then boils down to solve ordinary differential equations (3), bearing in mind that $\mathbf{E}$ is an unknown field.
In the sequel, we only consider the $1D$ case, which comes down to:

$$\begin{cases} \partial_t f + v \cdot \partial_x f + \mathrm{E} \cdot \partial_v f = 0, & \text{(4a)} \\ \partial_x \mathrm{E}(t, x) = \displaystyle\int_{\mathbb{R}} f(t, x, v)\,\mathrm{d}v - 1. & \text{(4b)} \end{cases}$$

The phase space $[0, 1] \times \mathbb{R}$ is uniformly discretized: $x_i = i\Delta x$ and $v_j = j\Delta v$.

Truncation As the following methods involve v-integral over the whole space, we assume that f vanishes outside $(-L, L)$ for L large enough. In practice, L is tuned to 10.

2.1 Direct method

A first method consists in using (3d) and a Strang time splitting for solving (3b)-(3a) for each couple (x_i, v_j) in the phase space. More precisely, the classical semi-Lagrangian method is a predictor-corrector strategy:

- Compute a prediction $\tilde{\mathrm{E}}^{n+1}$ for the electric field using a leapfrog scheme to solve the following equation deduced from (1) by integration with respect to $\boldsymbol{v}$:

$$\partial_t \left(\int_{\mathbb{R}^d} f(t, \boldsymbol{x}, \boldsymbol{v})\,\mathrm{d}\boldsymbol{v} \right) + \nabla_x \cdot \left(\int_{\mathbb{R}^d} \boldsymbol{v} f(t, \boldsymbol{x}, \boldsymbol{v})\,\mathrm{d}\boldsymbol{v} \right) = 0.$$

- Knowing $\tilde{\mathrm{E}}^{n+1}$, apply a backward resolution to (3).
- Once $\xi_{ij}^n \approx \mathcal{X}(t^n; t^{n+1}, x_i, v_j)$ and $\zeta_{ij}^n \approx \mathcal{V}(t^n; t^{n+1}, x_i, v_j)$ computed, use a cubic B-spline interpolation to evaluate $f(t^n, \xi_{ij}^n, \zeta_{ij}^n)$.

2.2 Splitting method

This method relies on a Strang x/v operator splitting:

- Resolution of $\partial_t f + v \cdot \partial_x f = 0$ starting from f^n over $[t^n, t^n + \Delta t/2]$; this step provides an approximation f^*. It is then possible to update the temporary electric field $\mathrm{E}^{n+1/2}$.
- Resolution of $\partial_t f + \mathrm{E}^{n+1/2} \cdot \partial_v f = 0$ starting from f^* over $[t^n, t^{n+1}]$; the resulting solution is denoted by f^{**}.
- Resolution of $\partial_t f + v \cdot \partial_x f = 0$ starting from f^{**} over $[t^n + \Delta t/2, t^{n+1}]$, which provides the solution at time t^{n+1}.

Each step in this algorithm may be solved exactly as the advection field is constant with respect to the spatial/velocity variable. An interpolation is however required.

2.3 Conservative method

In this method, we refer to the framework of finite volumes insofar as the unknowns are the mean values over cells of the solution:

$$f_i^n := \frac{1}{\Delta z} \int_{z_{i-1/2}}^{z_{i+1/2}} f(t^n, z)\,\mathrm{d}z.$$

The Strang operator splitting described in the previous method consists of resolutions of equations of the form:

$$\partial_t f + \partial_z (f \mathcal{E}_z) = 0,$$

where $z \in \{x, v\}$, $\mathcal{E}_x = v$ and $\mathcal{E}_v = \mathrm{E}(t, x)$ (due to the fact that $\mathcal{E}_z$ is independent from z). As the characteristic curves are here nothing but straight lines, we can explicitly express f^{n+1} as a function of f^n. Replacing f^{n+1} by its expression in f_i^{n+1} and applying a trivial change of variables, we derive this scheme:

$$f_i^{n+1} = f_i^n - \frac{\Phi_{i+1/2}^n - \Phi_{i-1/2}^n}{\Delta z}, \qquad \Phi_{i+1/2}^n = \int_{z_{i+1/2}}^{z_{i+1/2} - \Delta t\,\mathcal{E}_z} f(t^n, z)\,\mathrm{d}z. \tag{5}$$

It then remains to express $\Phi_{i+1/2}^n$ as a function of $(f_j^n)_j$, which is a matter of reconstruction. Let us note j the index of the cell where the foot $z_{i+1/2} - \Delta t\,\mathcal{E}_z$ of the characteristic curve is located. For $z \in [z_{j-1/2}, z_{j+1/2}]$ the solution is reconstructed by means of the formulae:

$$\tilde{f}_j^n(z) = f_j^n + \frac{z - z_j}{\Delta z} \times \begin{cases} \varepsilon_j^+ (f_{j+1}^n - f_j^n), & \text{if } \mathcal{E}_z > 0, \\ \varepsilon_j^- (f_j^n - f_{j-1}^n), & \text{if } \mathcal{E}_z < 0, \end{cases}$$

with $f_\infty^n = \max_k f_k^n$,

$$\varepsilon_j^+ = \begin{cases} \min\left(1, \dfrac{2f_j^n}{f_{j+1}^n - f_j^n}\right), & \text{if } f_{j+1}^n - f_j^n > 0, \\ \min\left(1, \dfrac{-2(f_\infty^n - f_j^n)}{f_{j+1}^n - f_j^n}\right), & \text{if } f_{j+1}^n - f_j^n < 0, \end{cases}$$

and

$$\varepsilon_j^- = \begin{cases} \min\left(1, \dfrac{2(f_\infty^n - f_j^n)}{f_j^n - f_{j-1}^n}\right), & \text{if } f_j^n - f_{j-1}^n > 0, \\ \min\left(1, \dfrac{-2f_j^n}{f_{j+1}^n - f_j^n}\right), & \text{if } f_j^n - f_{j-1}^n < 0. \end{cases}$$

This enables to compute $\Phi_{i+1/2}^n$.

1. Derive Formula (5).
2. Compute the numerical flux $\Phi_{i+1/2}^n$ for $f(t^n, z)$ approximated by $\tilde{f}_j^n(z)$ in both sign cases.

3 Directions

You may implement the three semi-lagrangian methods, write a report answering the questions and presenting relevant simulations to emphasize advantages and drawbacks of each method. Simulations may be carried out for the linear Landau damping in 1D (see reference given in the introduction). You are expected to deliver codes and report on **March 8., 2011** by email to:

`thierry.goudon@inria.fr` ; `sebastian.minjeaud@inria.fr` ; `yohan.penel@inria.fr`

A presentation session will be held on **March 11., 2011 at 3pm**. You will outline the methods and present striking results in 20 minutes.